Analytic equivalence relations with $\aleph_1$-many classes: A computability theoretic approach.

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for $f : \mathbb{R} \to \mathbb{R}$ $\quad \dfrac{\text{Borel}}{\text{continuous}} = \dfrac{\text{hyperarithmetic}}{\text{computable}} \quad$ for $f : \mathbb{N} \to \mathbb{N}$

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- $X \in L(\omega_1^{CK})$.
- $X = \{n \in \mathbb{N} : \varphi(n)\}$, where φ is a computable infinitary formula.

  (these are $L_{\omega_1,\omega}$ formulas where disjunctions and conjunctions are computable)

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Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Our main result

Definition:

- Given linear orderings $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *embeds in* $\mathcal{B}$ if there is a strictly increasing map $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.
- $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$. We write $\mathcal{A} \sim \mathcal{B}$.

Example:

$$\omega + \omega^* + \omega + \omega^* + \cdots \quad \sim \quad \omega^* + \omega + \omega^* + \omega + \cdots$$

Observation: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.

Proof: $\mathcal{L} \preccurlyeq \alpha \implies \mathcal{L}$ is an ordinal and $\mathcal{L} \leq \alpha$.
$\alpha \preccurlyeq \mathcal{L} \implies \alpha \leq \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$. □

Theorem: Every hyperarithmetic linear ordering is equimorphic to a recursive one.

A generalization to Linear orderigns

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The proof uses Laver's theorem on the well-quasi-orderness of linear orderings to analyze their structure under embeddability.

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The proof uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. It also uses that hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$.

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- isomorphism on models of a *counterexample to Vaught's conjecture* (relativized);

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The Main Theorem

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models. The following are equivalent:

- *T is a counterexample to Vaught's conjecture.*
- *T satisfies hyperarithmetic-is-recursive on a cone.*
- *There exists an oracle relative to which*

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} = \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$

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Theorem [Morley 70] The number of countable models of a theory T is either countable, $\aleph_1$, or $2^{\aleph_0}$.

Obs: This is a particular case of Bruges [78] that holds for all $\mathbf{\Sigma}^1_1$ equivalence relations.

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Obs: Since, in computability theory, almost all proofs relativize:
For "natural" classes of structures $\mathbb{K}$,

$\mathbb{K}$ *satisfies hyperarithmetic-is-recursive* $\iff$ *it does on a cone.*

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Def: A structure $\mathcal{A}$ is *computable (Muchnik reducible)* in a structure $\mathcal{B}$, if every presentation of $\mathcal{B}$ computes a presentation of $\mathcal{A}$.

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A sufficient condition for hyp-is-rec.

Def: For $\mathfrak{K} \subseteq 2^\omega$, $(\mathfrak{K}, \equiv, r)$ is a *ranked equivalence relation* if
$\equiv$ is an equivalence relation on $\mathfrak{K}$, and $r \colon \mathfrak{K}/\equiv \to \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *scattered* if
$r^{-1}(\alpha)$ contains countably many equivalence classes for each $\alpha \in \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *projective* if
$\mathfrak{K}$ and $\equiv$ are projective and r has a projective presentation $2^\omega \to 2^\omega$.

Theorem ([M.] (ZFC+PD))

Let $(\mathfrak{K}, \equiv, r)$ be scattered projective ranked equivalence relation such that $\forall Z \in \mathfrak{K},\ r(Z) < \omega_1^Z$.
For every X on a cone, (i.e. $\exists Y \forall X \geq_T Y$,) every equivalence class with an X-hyperarithmetic member has an X-computable member.

Lemma: [Martin] (ZFC+PD) If $f \colon 2^\omega \to \omega_1$ is projective and $f(X) < \omega_1^X$, then f is constant on a cone.

The main theorem

Theorem ([M. 13] (ZFC + ($0^\sharp$ exists) + $\neg$CH)

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This theorem applies to all the examples mentioned before.

Examples:

- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught's conjecture;
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In all the natural examples, the base of the cone is 0, which doesn't follow from the Theorem.

(May skip this) Removing ($0^\sharp$ exists) and ($\neg$CH)

Def: $\mathcal{S} \subseteq 2^{\mathbb{N}}$ is *cofinal (in the Turing degrees)* if $\forall Y\ \exists X \geq_T Y\ (X \in \mathcal{S})$.

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Theorem [M. 13] (ZF) The following are equivalent:

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Martin's Conjecture

Def: A *cone* is a set of the form $\{X \in 2^{\mathbb{N}} : X \geq_T Y\}$ for some $Y \in 2^{\mathbb{N}}$.

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Def: $\mathbb{K}$ is *on top* if for any Σ^1_1 equivalence relation E on $\mathbb{N}$, $E \leq_{eff} I_{\mathbb{N}}(\mathbb{K})$.

Some results about FF reductions

Theorem ([Fokina, Friedman, Harizanov, Knight, McCoy, M.])

The following classes of structures are on top (under $\leq_{eff}$):

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A *nice class of structures* $\mathbb{K}$ is one axiomatizable by an $L_{\omega_1,\omega}$ sentence.

One direction of the question.

Theorem ([Knight–M. 12; Becker 12])

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Obs: The theorem above implies the downward direction.

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Definition Let $2^{\alpha\circ} = \{\sigma \in 2^{\alpha} : \{\xi < \alpha : \sigma(\xi) = 1\}$ is finite $\}$.

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Theorem : [M.14] (ZFC+PD) If there exists $g\colon \omega_1 \to \omega_1$, such that there is a g-α-tree for $\mathbb{K}$ $(\forall \alpha < \omega_1)$, then $\mathbb{K}$ is on top on a cone.

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is a map $\sigma \mapsto \mathcal{A}_\sigma \colon 2^{\alpha\circ} \to \mathbb{K}$ such that

$$(\forall \sigma, \tau \in 2^{\alpha\circ})\ \ (\forall \xi < \alpha)\ \ \begin{cases} \sigma \restriction \xi = \tau \restriction \xi & \Rightarrow \mathcal{A}_\sigma \equiv_\xi \mathcal{A}_\tau \\ \sigma \restriction \xi \neq \tau \restriction \xi & \Rightarrow \mathcal{A}_\sigma \not\equiv_{g(\xi)} \mathcal{A}_\tau \end{cases}$$

Theorem : [M.14] (ZFC+PD) If there exists $g\colon \omega_1 \to \omega_1$, such that there is a g-α-tree for $\mathbb{K}$ $(\forall \alpha < \omega_1)$, then $\mathbb{K}$ is on top on a cone.

Theorem : [M.14] (ZFC+PD) If $\mathbb{K}$ is on top, there is a Δ^1_1 $g\colon \omega_1^{CK} \to \omega_1^{CK}$ such that there is a computable g-α-tree for $\mathbb{K}$ $(\forall \alpha < \omega_1^{CK})$.

The Main results

Theorem ([M. 12] (ZFC+$0^\sharp$ exists))

Let E be an analytic equivalence relation. The following are equivalent:

- *E does not have perfectly many classes.*
- *E satisfies hyperarithmetic-is-recursive on a cone.*
- *The classes of E are linearly ordered by Muchnik reducibility.*

Theorem [M. 15] Let $\sim$ be a degree-invariant, $\mathbf{\Sigma}^1_1$ equivalence relation. Exactly one of the following holds:

- $\sim$ has perfectly many classes on every cone.
- $\sim$ is trivial on a cone (i.e., $X \sim Y$ for all X, Y on some cone).
- $X \sim Y \iff \omega_1^X = \omega_1^Y$ for every X, Y on some cone.

Open question: Are the following equivalent?

- No theory is intermediate for effective reducibility on a cone.
- Vaught's conjecture.