Convexly valued o-minimal fields

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O-minimality

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This condition on one-variable definable sets has strong consequences for definable sets in higher dimensions. Perhaps most prominently, one has a cell decomposition theorem. A consequence of cell decomposition is that o-minimality is really strong o-minimality.

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- vast generalizations thereof

Valuations

We let R be an o-minimal field (i.e. an o-minimal expansion of a real closed field) and V a convex subring (for example, the convex hull of $\mathbb{Q}$ in R). Then V is in particular a valuation ring, i.e. it has a unique maximal ideal $\mathfrak{m}$.

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$v : R^\times \to R^\times/V^\times$, where $\Gamma := R^\times/V^\times$ is the value group.

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- Limit sets: One can use valuations to show that in o-minimal expansions of $\overline{\mathbb{R}}$, Hausdorff limits of definable families form definable families (see for example [4]).
- Preparation theorems: Prepared functions of several variables depend in a piecewise simple way on any chosen variable. The existence of prepared versions of definable functions in certain o-minimal structures can be viewed as a geometric translation of valuation theoretic facts (see for example [6]).

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For o-minimal fields, a good analogue of convex subrings of real closed fields are T-convex subrings (van den Dries, Lewenberg [5]). The T-convex subrings of R are precisely the convex hulls of the elementary substructures of R.

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Among the nice properties of T-convex structures are quantifier elimination and o-minimality of the residue field (with induced structure) – in fact one has $Th(\mathbf{k}) = Th(R)$.

o-minimal residue fields

T-convexity does not capture all cases of interest. For example, if V is the convex hull of $\mathbb{Q}$ in R, then V is not necessarily T-convex (the language of R might contain a constant symbol for an element in $R^{>V}$, or $Th(R)$ might not be pseudo-real).

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We shall consider (R, V) such that $\mathbf{k}$ with induced structure is o-minimal. This does not only include all cases where V is the convex hull of $\mathbb{Q}$ in R, but also all instances in which V is T-convex.

Some results on (R, V) with o-minimal residue field

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Theorem (M.)

$\mathbf{k}$ is o-minimal iff for each definable $f : R \to R$ there is $\epsilon_0 \in \mathfrak{m}^{>0}$ so that $\operatorname{res} f(\epsilon_0) = \operatorname{res} f(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$.

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The above condition is equivalent to:

Whenever $Y \subseteq \mathbf{k}^n$ is closed and definable in $\mathbf{k}$ with its induced structure, then there is $X \subseteq R^n$ definable in R such that $\operatorname{res} X = Y$.

QE for (R, V)

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The theorem follows by a short, elementary proof from a model-completeness result in [7].

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The proof of model completeness uses (somewhat surprisingly) abstractly model-theoretic notions such as Morley sequences and dividing. An essential ingredient is the notion of separation as introduced by Baisalov and Poizat.

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Theorem (Ealy, M)

Let $R \preceq \mathcal{R}$, *let* $a \in \mathcal{R}$, *and let* $W \subseteq R\langle a \rangle$ *be such that* $(R, V) \subseteq (R\langle a \rangle, W)$. *Then* $(R, V) \preceq (R\langle a \rangle, W)$ *iff there there are no* R*-definable functions* f, g *such that* $f(a) \in W$, $g(a) > W$ *and* $V < f(a), g(a) < R^{>V}$.

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Next, one proves that the residue field is stably embedded; this can in turn be used to show that the above criterion for elementary extensions is satisfied whenever $(R, V) \subseteq (R', V')$ and $(R, V) \equiv (R', V')$.

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If $Y \subseteq \mathbf{k}^n$ is $\emptyset$-definable in the residue field, then there is $X \subseteq R^n$, $\emptyset$-definable in R, such that $\mathrm{res}(X) = Y$.

Substructures are elementary

Quantifier elimination then follows by establishing that substructures are elementary.

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Lemma

Let $a \in R$, and let $V_a = V \cap R_0\langle a\rangle$. Then

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Corollary

$Th(R, V)$ is universally axiomatizable.

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$Th(R, V)$ has definable Skolem functions.

Open questions

- What is a universal axiomatization for $Th(R, V)$?

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- Do we have model completeness/quantifier elimination in a language without constants for elements of R_0?

Bibliography

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Bibliography

Y. Baisalov, B. Poizat, *Paires de Structures O-Minimales.* J. Symb. Log. 63(2): 570-578 (1998).

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J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures II*, Trans. AMS 295 (1986) 593-605.

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- J. Maříková, *O-minimal fields with standard part map*. Fund. Math. 209 (2010), no. 2, 115-132.
- J. Maříková *O-minimal residue fields of o-minimal fields* Ann. Pure Appl. Logic 162 (2011), no. 6, 457-464.

Bibliography

Y. Baisalov, B. Poizat, *Paires de Structures O-Minimales.* J. Symb. Log. 63(2): 570-578 (1998).

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L. van den Dries, P. Speissegger, *O-minimal preparation theorems.* Model theory and applications, 87â116, Quad. Mat., 11, Aracne, Rome, 2002.

C. F. Ealy, J. Maříková, *Model completeness of o-minimal fields with convex valuations.* The Journal of Symbolic Logic, 80(1), 234-250, 2015.

J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures II*, Trans. AMS 295 (1986) 593-605.

J. Maříková, *O-minimal fields with standard part map*. Fund. Math. 209 (2010), no. 2, 115-132.

J. Maříková *O-minimal residue fields of o-minimal fields* Ann. Pure Appl. Logic 162 (2011), no. 6, 457-464.

A. Pillay and C. Steinhorn *Definable sets in ordered structures I.*, Trans. AMS 295 (1986), 565-592.