Recursion Theoretic Methods in Descriptive Set Theory and Infinite Dimensional Topology

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Joint Work with

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History of Borel Isomorphism Problem

1. (Kuratowski 1934) There is only one uncountable Polish space up to Borel isomorphism.
2. (Harrington, Steel, 1970s) The following are equivalent:
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Definition

We say that X is α-th level Borel isomorphic to Y if $(X, \utilde{\mathbf{\Sigma}}^0_{1+\alpha}(X)) \simeq (Y, \utilde{\mathbf{\Sigma}}^0_{1+\alpha}(Y))$, i.e., there is a bijection between X and Y preserving the Borel hierarchy above $\mathbf{\Sigma}^0_{1+\alpha}$.

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homeomorphism $=$ $\mathbf{0}$-th level Borel isomorphism
$\Rightarrow$ α-th level Borel isomorphism $\Rightarrow$ $(\alpha + \mathbf{1})$-th level Borel isomorphism
$\Rightarrow$ Borel isomorphism

How many Polish spaces are there up to α-th level Borel isomorphism?

Theorem

Let X and Y be uncountable Polish spaces.

1. (Kuratowski) There is only one uncountable Polish space up to α-th level Borel isomorphism for any $\alpha \geq \omega$.
2. (Jayne, 1970s) If X is first-level Borel isomorphic to Y, i.e., $(X, F_\sigma(X)) \simeq (Y, F_\sigma(Y))$, then $\dim(X) = \dim(Y)$.
3. (Jayne-Rogers, 1970s) If X is the union of countably many finite dimensional subspaces, then X is second-level Borel isomorphic to $\mathbb{R}$, i.e., $(X, G_{\delta\sigma}(X)) \simeq (\mathbb{R}, G_{\delta\sigma}(\mathbb{R}))$.
4. $\mathbb{R}$ is not finite-level Borel isomorphic to $[0,1]^{\mathbb{N}}$.

How many Polish spaces are there up to α-th level Borel isomorphism?

- There are continuum many Polish spaces up to first level Borel isomorphism
- There are at least two Polish spaces up to n-th level Borel isomorphism for any $n < \omega$
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Second Level Borel Isomorphism Problem

Is there a third Polish space up to second-level Borel isomorphism?

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Second Level Borel Isomorphism Problem

Is there a third Polish space up to second-level Borel isomorphism?

- An invariant which we call *degree co-spectrum*, a collection of Turing ideals realized as lower Turing cones of points of a Polish space, plays a key role.
- The key idea is measuring the quantity of all possible Scott ideals (ω-models of $\mathbf{WKL_0}$) realized within the degree co-spectrum (on a cone) of a given space.

Let $\mathcal{B}^*_\alpha(X)$ be the Banach algebra of bounded real valued Baire class α functions on X w.r.t. the supremum norm and pointwise operation.

Background in Banach Space Theory

- The basic theory on the Banach spaces $\mathcal{B}^*_\alpha(X)$ has been studied by Bade, Dachiell, Jayne and others in 1970s.
- Jayne (1974) proved an analogue of the *Banach-Stone Theorem* and the *Gel'fand-Kolmogorov Theorem* for Baire classes, that is, the α-th level Baire structure of a space X is determined by the ring structure of the Banach algebra $\mathcal{B}^*_\alpha(X)$, and vice versa.

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Theorem (Jayne 1974)

The following are equivalent for realcompact spaces X and Y:

1. X is α-th level Baire isomorphic to Y.
2. $\mathcal{B}^*_\alpha(X)$ is linearly isometric to $\mathcal{B}^*_\alpha(Y)$.
3. $\mathcal{B}^*_\alpha(X)$ is ring isomorphic to $\mathcal{B}^*_\alpha(Y)$.

Main Problem (Motto Ros)

Suppose that X is an uncountable Polish space.
Is the Banach algebra $\mathcal{B}^*_n(X)$ linearly isometric (ring isomorphic) to either $\mathcal{B}^*_n(\mathbb{R})$ or $\mathcal{B}^*_n(\mathbb{R}^{\mathbb{N}})$ for some $n \in \omega$?

- By Jayne's theorem (1974), Motto Ros' problem is equivalent to Second Level Borel Isomorphism Problem.
- Any counterexample of this problem must be infinite-dimensional.

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- At that time, almost no nontrivial proper infinite dimensional Polish spaces had been discovered yet.
- Perhaps, it had been expected that the structure of proper infinite dim. Polish spaces is simple — this conclusion was too hasty!
- By using Computability Theory, we reveal that the second level Borel isomorphic classification of Polish spaces is highly nontrivial!

Main Theorem (K. and Pauly)

There exists a $2^{\aleph_0}$ collection $(X_\alpha)_{\alpha<2^{\aleph_0}}$ of topological spaces s.t.

1. X_α is an infinite dimensional Cantor manifold for any $\alpha < 2^{\aleph_0}$, i.e., X_α is compact metrizable, and if $X_\alpha \setminus C = U_1 \sqcup U_2$ for some nonempty open U_1, U_2, then C must be infinite dimensional.

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2. X_α possesses Haver's property C (hence, weakly infinite dimensional) for any $\alpha < 2^{\aleph_0}$.
3. If $\alpha \neq \beta$, then X_α is not n-th level Borel isomorphic to X_β.
4. If $\alpha \neq \beta$, then the Banach algebra $\mathcal{B}^*_n(X_\alpha)$ is not linearly isometric (not ring isomorphic etc.) to $\mathcal{B}^*_n(X_\beta)$ for any $n \in \omega$.

Decomposition Theorem (K.; Gregoriades and K.; K. and Ng)

If $f : X \to Y$ is a function from analytic sp. X into Polish sp. Y s.t.

$$A \subseteq \underset{\sim}{\Sigma}^0_{m+1}(Y) \;\Rightarrow\; f^{-1}[A] \in \underset{\sim}{\Sigma}^0_{n+1}(X)$$

then, there exists a countable partition $(X_i)_{i\in\omega}$ of X such that the restriction $f|_{X_i}$ is $\underset{\sim}{\Sigma}^0_{n-m+1}$-measurable for every $i \in \omega$.

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- We then have $(f(x) \oplus z)^{(m)} \leq_T (x \oplus (z \oplus p)^{(\xi)})^{(n)}$ for all $z \in 2^\omega$, where $\leq_T$ is generalized Turing reducibility on represented spaces.

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- By the Shore-Slaman join theorem for any Polish degree structure, we have $f(x) \leq_T (x \oplus p^{(\xi)})^{(n-m)}$.
- Therefore, f is decomposed into countably many $\underset{\sim}{\Sigma}^0_{n-m+1}$-measurable functions $x \mapsto \Phi_e((x \oplus p^{(\xi)})^{(n-m)})$, $e \in \omega$.

The role of the Decomposition Theorem here is for showing that every n-th Borel isomorphism is covered by ω-many partial homeomorphisms.

$X \preceq_{pw} Y$ means that there is a countable cover $\{X_i\}_{i\in\omega}$ of X s.t. X_i is topologically embedded into Y for every $i \in \omega$.

Main Problem

Does there exist an uncountable Polish space X satisfying either of the following equivalent conditions?

1. $B_2^*(X)$ is linearly isometric neither to $B_2^*(\mathbb{R})$ nor to $B_2^*(\mathbb{R}^{\mathbb{N}})$.
2. $B_2^*(X)$ is ring isomorphic neither to $B_2^*(\mathbb{R})$ nor to $B_2^*(\mathbb{R}^{\mathbb{N}})$.
3. X is 2^{nd} level Borel isomorphic neither to $\mathbb{R}$ nor to $\mathbb{R}^{\mathbb{N}}$.
4. $\mathbb{R} \prec_{pw} X \prec_{pw} \mathbb{R}^{\mathbb{N}}$.

Compared to the Borel isomorphism problem in 1970s:

- The *Borel isomorphism problem* on analytic spaces was able to be reduced to the same problem on *zero-dimensional* analytic spaces.
- The *second-level Borel isomorphism problem* is inescapably tied to *infinite dimensional* topology.

- Recall: Jayne-Rogers (1979) showed that any two uncountable Polish spaces that are countable unions of sets of finite dimension are $\mathbf{2}^{nd}$-level Borel isomorphic.
- Indeed, Hurewicz-Wallman (1941) showed that

$$X \simeq_{pw} \mathbb{R} \iff \mathbf{trind}(X) < \infty,$$

where **trind** is transfinite inductive dimension.

- (Alexandrov 1948) X is *weakly infinite dimensional* (w.i.d.) if for each sequence (A_i, B_i) of pairs of disjoint closed sets in X there are separations L_i in X of A_i and B_i s.t. $\bigcap_i L_i = \emptyset$.
- (Haver 1973, Addis-Gresham 1978) X is a C-space ($S_c(\mathcal{O},\mathcal{O})$) if for each sequence $(\mathcal{U}_i)$ of open covers of X there is a pairwise disjoint open family $(\mathcal{V}_i)$ refining $(\mathcal{U}_i)$ s.t. $\bigcup_i \mathcal{V}_i$ covers X.

$$X \leq_{pw} 2^{\mathbb{N}} \;\Leftrightarrow\; \mathrm{trind}(X) < \infty \;\Rightarrow\; X \text{ is } C \;\Rightarrow\; X \text{ is w.i.d.}$$

- (Alexandrov 1951) $\exists$ a w.i.d. metrizable compactum $X >_{pw} 2^{\mathbb{N}}$?
- (R. Pol 1981) There exists a metrizable C-compactum $X >_{pw} 2^{\mathbb{N}}$.
- (E. Pol 1997) There exists an infinite dimensional C-Cantor manifold, i.e., a C-compactum which cannot be separated by any hereditarily weakly infinite dimensional closed subspaces.
- (Chatyrko 1999) There is a collection $\{X_\alpha\}_{\alpha<2^{\aleph_0}}$ of continuum many infinite dimensional C-Cantor manifolds such that X_α cannot be embedded into X_β whenever $\alpha \neq \beta$.

An *infinite dimensional* $\boldsymbol{C}$*-Cantor manifold* is a $\boldsymbol{C}$-compactum which cannot be separated by any hereditarily weakly infinite dimensional closed subspace.

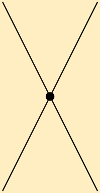
Main Lemma (K. and Pauly)

Let $\mathfrak{M}_\infty$ be the class of all infinite dimensional $\boldsymbol{C}$-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^\omega, \subseteq)$ into $(\mathfrak{M}_\infty, \preceq_{pw})$.

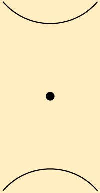
- This solves Motto Ros' problem (and the second level Borel isomorphism problem) in Banach Space Theory.
- This strengthen R. Pol's theorem and Chatyrko's theorem in Infinite Dimensional Topology.

To show Main Lemma, we again use Computability Theory!

Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces



(a) Any point in $\mathbb{R}^{\boldsymbol{n}}$

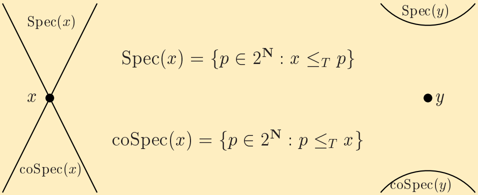


(b) Some point in $\mathbf{[0,1]}^{\mathbb{N}}$

Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces

(a) Any point in $\mathbb{R}^n$ (b) Some point in $[0,1]^{\mathbb{N}}$

- By *approximating* each point in a space X *by a zero-dim space*, we measure "*how similar the space X is to a zero-dim space*".
- (a) Upper and lower approximations by a zero-dim space *meet*.
- (b) There is a *gap* between upper and lower approximations by a zero-dim space



(a) Any point in $\mathbb{R}^n$ (b) Some point in $[0,1]^{\mathbb{N}}$

- $\mathrm{Spec}(x) = \{p \in 2^{\mathbb{N}} : x \leq_T p\}$.
- $\mathrm{coSpec}(x) = \{p \in 2^{\mathbb{N}} : p \leq_T x\}$.

Key Idea

Classification of topological spaces by degrees of unsolvability:

1. The Turing degrees $\simeq$ the degree structure on Cantor space $\mathbf{2}^{\mathbb{N}}$ and Euclidean spaces $\mathbb{R}^n$.
2. The enumeration degrees $\simeq$ the degree structure on the Scott domain $\mathcal{P}(\mathbb{N})$.
3. Hinman (1973): degrees of unsolvability of continuous functionals $\simeq$ the degree structure on the space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ of Kleene-Kreisel continuous functionals.
4. J. Miller (2004): continuous degrees $\simeq$ the degree structure on the function space $\mathcal{C}([0,1])$ and the Hilbert cube $[0,1]^{\mathbb{N}}$.

Definition

Let X and Y be second-countable T_0 spaces with fixed countable open basis $\{B_n^X\}_{n\in\omega}$ and $\{B_n^Y\}_{n\in\omega}$.
A point $x \in X$ is "*Turing reducible*" to a point $y \in Y$ ($x \leq_T y$) if

$$\{n \in \omega : x \in B_n^X\} \leq_e \{n \in \omega : y \in B_n^Y\}.$$

In other words, we identify the "*Turing degree*" of $x \in X$ with the *enumeration degree of the (coded) neighborhood filter* of x.

Example

- The degree structure of *Cantor space* is exactly the same as the *Turing degrees*.
- The degree structure of *Hilbert cube* (a universal Polish space) is exactly the same as the *continuous degrees*.
- The degree structure of *the Scott domain* $\mathcal{O}(\mathbb{N})$ (a universal quasi-Polish space) is exactly the same as the *enumeration degrees*.

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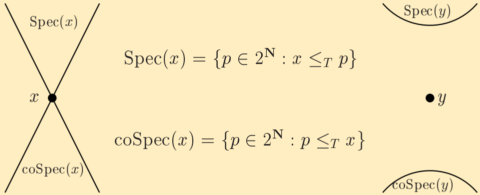

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Lemma (K. and Pauly)

$X \simeq_{pw} Y \Longrightarrow \mathbf{Spec}^r(X) = \mathbf{Spec}^r(Y)$ for some oracle $r \in 2^\omega$.
$\Longrightarrow \mathbf{coSpec}^r(X) = \mathbf{coSpec}^r(Y)$ for some oracle $r \in 2^\omega$.

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$\Longrightarrow \mathbf{coSpec}^r(X) = \mathbf{coSpec}^r(Y)$ for some oracle $r \in 2^\omega$.

1. A Turing ideal $\mathcal{J} \subseteq 2^\omega$ is *realized* by x if $\mathcal{J} = \mathrm{coSpec}(x)$.
2. A countable set $\mathcal{J} \subseteq 2^\omega$ is a *Scott ideal*
$\iff (\omega, \mathcal{J}) \models \mathbf{RCA} + \mathbf{WKL}$.

Realizability of Scott ideals (J. Miller 2004)

1. $2^\omega \simeq_{pw} \omega^\omega \simeq_{pw} \mathbb{R}^n \simeq_{pw} \bigoplus_{n\in\omega} \mathbb{R}^n$. (*Turing degrees*.)
No Scott ideal is realized in these spaces!
2. $[0,1]^\omega \simeq_{pw} \mathcal{C}([0,1]) \simeq_{pw} \ell^2$. (*full continuous degrees*.)
Every countable Scott ideal is realized in these spaces!

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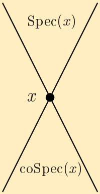

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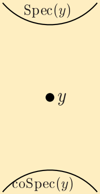

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- **Spec** determines the pw-homeomorphism type of a space, and **coSpec** is invariant under pw-homeomorphism.
- The **coSpec** of any point in a space of **dim** $< \infty$ has to be a principal Turing ideal.
- (Miller) Every countable Scott ideal is realized as **coSpec** of a point in Hilbert cube.

Definition

$\Gamma : 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *ω-left-CEA operator* if the infinite sequence $\Gamma(y) = (x_0, x_1, x_2, \dots)$ is generated in a uniformly left-computably enumerable manner by a single Turing machine, that is, there is a left-c.e. operator γ such that for all i,

$$x_i := \Gamma(y)(i) = \gamma(y, i, x_0, x_1, \dots, x_{i-1}).$$

An ω-left-CEA operator $\Gamma : \mathbb{N} \times 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *universal* if for every ω-left-CEA operator Ψ, there is e such that $\Psi = \lambda y.\Gamma(e, y)$.

Let ω**CEA** denote the graph of a universal ω-left-CEA operator.

Theorem (K.-Pauly)

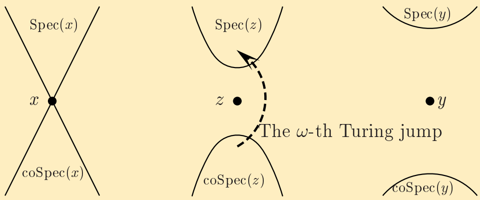
The space ω**CEA** (as a subspace of Hilbert cube) is an intermediate Polish space:

$$\mathbf{2}^{\mathbb{N}} \prec_{pw} \omega\mathbf{CEA} \prec_{pw} [\mathbf{0},\mathbf{1}]^{\mathbb{N}}$$

Remark

Furthermore, ω**CEA** is pw-homeomorphic to the following:

- Rubin-Schori-Walsh (1979)'s strongly infinite dimensional totally disconnected Polish space.
- Roman Pol (1981)'s weakly infinite dimensional compactum which is not decomposable into countably many finite-dim subspaces (a solution to Alexandrov's problem).



(a) $\mathbf{2}^{\mathbb{N}}$ (b) ω**CEA** (c) $[\mathbf{0}, \mathbf{1}]^{\mathbb{N}}$

- (a) **coSpec** is principal, and *meets* with **Spec**.
- (b) **coSpec** is not always principal, but the "*distance*" between **Spec** and **coSpec** has to be at most *the ω-th Turing jump*.
- (c) **coSpec** can realize an arbitrary countable Scott ideal, hence **Spec** and **coSpec** can be separated by *an arbitrary distance*.

$$\omega\mathbf{CEA} = \{(e, p, x_0, x_1, \dots) \in \omega \times 2^\omega \times [0,1]^\omega : (\forall i)\ x_i \text{ is the } e\text{-th left-c.e. real in } (p, x_0, x_1, \dots, x_{i-1}).\}$$

Lemma

For any $p \in 2^\omega$, the following Scott ideal is not realized in ω**CEA**:

$$\mathcal{J}^p = \{z \in 2^\omega : (\exists n)\ z \leq_T p^{(\omega \cdot n)}\}.$$

- Pick $z = (e, p, x_0, x_1, \dots) \in \omega\mathbf{CEA}$.
- Then, $p \in \mathbf{coSpec}(z)$ and $p^{(\omega)} \in \mathbf{Spec}(z)$.
- Clearly, $p^{(\omega+1)} \notin \mathbf{coSpec}(z)$.

Since **coSpec** (up to an oracle) is invariant under pw-homeomorphism, we have $\omega\mathbf{CEA} \prec_{pw} [0,1]^{\mathbb{N}}$.

Another separation is based on Kakutani's fixed point theorem.

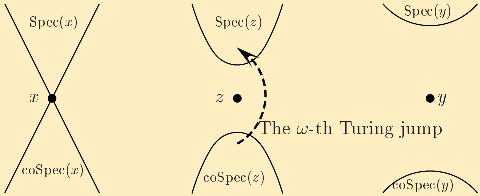
Theorem (J. Miller 2004)

There is a nonempty convex-valued computable function $\Psi : [0,1]^{\mathbb{N}} \to \mathcal{P}([0,1]^{\mathbb{N}})$ with a closed graph such that for every fixed point $\langle x_0, x_1, \dots \rangle \in \mathrm{Fix}(\Psi)$,

$$\mathrm{coSpec}(\langle x_0, x_1, x_2, \dots \rangle) = \{x_0, x_1, x_2, \dots\}.$$

Moreover, such an x realizes a Scott ideal.

- $\mathrm{Fix}(\Psi)$ is a Π^0_1 subset of $[0,1]^\omega$.
- Inductively find $(x_0, x_1, \dots) \in \mathrm{Fix}(\Psi)$, where x_{i+1} is the "*leftmost*" value s.t. $(x_0, x_1, \dots, x_{i+1})$ is extendible in $\mathrm{Fix}(\Psi)$.
- Then, x_{i+1} is left-c.e. in $(x_0, x_1, \dots, x_i)$, uniformly.
- x_{i+1} does not depend on the choice of a name of $(x_0, \dots, x_i)$.



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- (a) **coSpec** is principal, and *meets* with **Spec**.
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1. $\mathbf{coSpec}(2^{\mathbb{N}}) =$ all principal Turing ideals.
2. $\mathbf{coSpec}([0,1]^{\mathbb{N}}) =$ all principal Turing ideals and Scott ideals.
3. What do we know about $\mathbf{coSpec}(\omega\mathbf{CEA})$?
   - It cannot realize an ω-jump ideal.
   - It realizes a non-principal Turing ideal.
   - We know absolutely nothing about what kind of Turing ideals it realizes; even whether it realizes a jump ideal or not.

How can we control **coSpec** of a Polish space?

For instance, given $\alpha << \beta < \omega_1$, we need a technique for constructing a Polish space such that

- it cannot realize a β-jump ideal,
- it realizes an α-jump ideal.

We say that $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *oracle Π^0_2 singleton* if it has a Π^0_2 graph.
For instance, the α-th Turing jump operator TJ^{α} is an oracle Π^0_2 singleton.

Definition (Modified ωCEA Space)

The space $\omega\mathbf{CEA}(g)$ consists of $(d, e, r, x) \in \mathbb{N}^2 \times 2^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ such that for every i,

1. either $x_i = g^i(r)$, or
2. there are $u \le v \le i$ such that
   $x_i \in [0,1]$ is the e-th left-c.e. real in $\langle r, x_{<i}, x_{l(u)} \rangle$
   and $x_{l(u)} = g^{l(u)}(r)$, where $l(u) = \Phi_d(u, r, x_{<v})$.

Here: $g^0(x) = x$ and $g^{n+1}(x) = g^n(x) \oplus g(g^n(x))$.

We define $\mathbf{Rea}(g) = \omega\mathbf{CEA}(g) \cap (\mathbb{N}^2 \times \mathbf{Fix}(\Psi))$.
The subspace $\mathbf{Rea}(g)$ (as a subspace of $[0,1]^{\mathbb{N}}$) is Polish whenever g is an oracle Π^0_2 singleton.

Suppose that g is an oracle Π^0_2-singleton. For every oracle $r \in 2^{\mathbb{N}}$, consider two Turing ideals defined as

$$\mathcal{J}_T(g,r) = \{z \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_T g^n(r)\},$$
$$\mathcal{J}_a(g,r) = \{z \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_a g^n(r)\}.$$

Here: $\leq_a$ is the arithmetical reducibility.

Main Lemma (**coSpec**-Controlling)

1. For every $x \in \mathbf{Rea}(g)$, there is $r \in 2^{\mathbb{N}}$ such that
$$\mathcal{J}_T(g,r) \subseteq \mathbf{coSpec}(x) \subseteq \mathcal{J}_a(g,r).$$
2. For every $r \in 2^{\mathbb{N}}$, there is $x \in \mathbf{Rea}(g)$ such that
$$J_T(g,r) \subseteq \mathbf{coSpec}(x).$$

If $g = \mathbf{TJ}^\alpha$ is the α-th Turing jump operator for $\alpha \geq \omega$,

1. $\mathbf{coSpec}(\mathbf{Rea}(\mathbf{TJ}^\alpha))$ realizes no β-jump ideal for $\beta \geq \alpha \cdot \omega$,
2. $\mathbf{coSpec}(\mathbf{Rea}(\mathbf{TJ}^\alpha))$ realizes an α-jump ideal.

1. By **coSpec**-Controlling Lemma, given an oracle $\mathbf{\Pi}^0_2$ singleton g we can construct a Polish space which realizes all Turing ideals closed under g.
2. $\mathbf{Rea}(g)$ is strongly infinite dimensional and totally disconnected.
3. Hence, its compactification $\gamma\mathbf{Rea}(g)$ (in the sense of Lelek) is a "*Pol-type space*", hence, a metrizable C-compacta.
4. Note that Lelek's compactification preserves **Spec** and **coSpec**.
5. By combining Elzbieta Pol's construction, our spaces can be assumed to be infinite dimensional C-Cantor manifolds.

Main Lemma (K. and Pauly)

Let $\mathfrak{M}_\infty$ be the class of all infinite dimensional C-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^\omega, \subseteq)$ into $(\mathfrak{M}_\infty, \leq_{pw})$.

Main Theorem (K. and Pauly)

There exists a $2^{\aleph_0}$ collection $(X_\alpha)_{\alpha<2^{\aleph_0}}$ of topological spaces s.t.

1. X_α is an infinite dimensional Cantor manifold for any $\alpha < 2^{\aleph_0}$,
2. X_α possesses Haver's property C for any $\alpha < 2^{\aleph_0}$.
3. If $\alpha \neq \beta$, then X_α is not n-th level isomorphic to X_β for any $n \in \omega$.
4. If $\alpha \neq \beta$, then the Banach space $\mathcal{B}_n(X_\alpha)$ is not linearly isometric to $\mathcal{B}_n(X_\beta)$ for any $n \in \omega$.

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