Recursion Theoretic Methods in Descriptive Set Theory and Infinite Dimensional Topology

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History of Borel Isomorphism Problem

1. (Kuratowski 1934) There is only one uncountable Polish space up to Borel isomorphism.

2. (Harrington, Steel, 1970s) The following are equivalent:
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   2. There are exactly two uncountable analytic space up to Borel isomorphism.
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Definition

We say that \( X \) is \( \alpha \)-th level Borel isomorphic to \( Y \) if \((X, \Sigma^0_{1+\alpha}(X)) \simeq (Y, \Sigma^0_{1+\alpha}(Y))\), i.e., there is a bijection between \( X \) and \( Y \) preserving the Borel hierarchy above \( \Sigma^0_{1+\alpha} \).
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homeomorphism $= 0$-th level Borel isomorphism
$\Rightarrow \alpha$-th level Borel isomorphism $\Rightarrow (\alpha + 1)$-th level Borel isomorphism
$\Rightarrow$ Borel isomorphism
How many Polish spaces are there up to $\alpha$-th level Borel isomorphism?

**Theorem**

Let $X$ and $Y$ be uncountable Polish spaces.

1. (Kuratowski) There is only one uncountable Polish space up to $\alpha$-th level Borel isomorphism for any $\alpha \geq \omega$.

2. (Jayne, 1970s) If $X$ is first-level Borel isomorphic to $Y$, i.e., $(X, F_{\sigma}(X)) \simeq (Y, F_{\sigma}(Y))$, then $\dim(X) = \dim(Y)$.

3. (Jayne-Rogers, 1970s) If $X$ is the union of countably many finite dimensional subspaces, then $X$ is second-level Borel isomorphic to $\mathbb{R}$, i.e., $(X, G_{\delta\sigma}(X)) \simeq (\mathbb{R}, G_{\delta\sigma}(\mathbb{R}))$.

4. $\mathbb{R}$ is not finite-level Borel isomorphic to $[0, 1]^\mathbb{N}$.
How many Polish spaces are there up to $\alpha$-th level Borel isomorphism?

- There are **continuum many** Polish spaces up to first level Borel isomorphism.
- There are **at least two** Polish spaces up to $n$-th level Borel isomorphism for any $n < \omega$.
- There is **only one** Polish space up to $\alpha$-th level Borel isomorphism for any $\alpha \geq \omega$.
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Second Level Borel Isomorphism Problem

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Second Level Borel Isomorphism Problem

Is there a third Polish space up to second-level Borel isomorphism?

- An invariant which we call *degree co-spectrum*, a collection of Turing ideals realized as lower Turing cones of points of a Polish space, plays a key role.
- The key idea is measuring the quantity of all possible *Scott ideals* ($\omega$-models of $\text{WKL}_0$) realized within the degree co-spectrum (on a cone) of a given space.
Let $\mathcal{B}_\alpha^*(X)$ be the Banach algebra of bounded real valued Baire class $\alpha$ functions on $X$ w.r.t. the supremum norm and pointwise operation.

### Background in Banach Space Theory

- The basic theory on the Banach spaces $\mathcal{B}_\alpha^*(X)$ has been studied by Bade, Dachiell, Jayne and others in 1970s.
- Jayne (1974) proved an analogue of the **Banach-Stone Theorem** and the **Gel’fand-Kolmogorov Theorem** for Baire classes, that is, the $\alpha$-th level Baire structure of a space $X$ is determined by the ring structure of the Banach algebra $\mathcal{B}_\alpha^*(X)$, and vice versa.
Let $B^*_\alpha(X)$ be the Banach algebra of bounded real valued Baire class $\alpha$ functions on $X$ w.r.t. the supremum norm and pointwise operation.

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**Theorem (Jayne 1974)**

The following are equivalent for realcompact spaces $X$ and $Y$:

1. $X$ is $\alpha$-th level Baire isomorphic to $Y$.
2. $B^*_\alpha(X)$ is linearly isometric to $B^*_\alpha(Y)$.
3. $B^*_\alpha(X)$ is ring isomorphic to $B^*_\alpha(Y)$.
Main Problem (Motto Ros)
Suppose that $X$ is an uncountable Polish space. Is the Banach algebra $B^*_n(X)$ linearly isometric (ring isomorphic) to either $B^*_n(\mathbb{R})$ or $B^*_n(\mathbb{R}^\mathbb{N})$ for some $n \in \omega$?

- By Jayne’s theorem (1974), Motto Ros’ problem is equivalent to Second Level Borel Isomorphism Problem.
- Any counterexample of this problem must be infinite-dimensional.
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- At that time, almost no nontrivial proper infinite dimensional Polish spaces had been discovered yet.
- Perhaps, it had been expected that the structure of proper infinite dim. Polish spaces is simple — this conclusion was too hasty!
- By using *Computability Theory*, we reveal that the second level Borel isomorphic classification of Polish spaces is highly nontrivial!
Main Theorem (K. and Pauly)

There exists a $2^\aleph_0$ collection $(X_\alpha)_{\alpha<2^\aleph_0}$ of topological spaces s.t.

1. $X_\alpha$ is an infinite dimensional Cantor manifold for any $\alpha < 2^\aleph_0$, i.e., $X_\alpha$ is \textit{compact metrizable}, and if $X_\alpha \setminus C = U_1 \sqcup U_2$ for some nonempty open $U_1, U_2$, then $C$ must be infinite dimensional.

2. $X_\alpha$ possesses Haver's property $C$ (hence, weakly infinite dimensional).

3. If $\alpha, \beta$, then $X_\alpha$ is \textit{not} $n$-th level Borel isomorphic to $X_\beta$.

4. If $\alpha, \beta$, then the Banach algebra $B^*_n(X_\alpha)$ is \textit{not} linearly isometric (not ring isomorphic etc.) to $B^*_n(X_\beta)$ for any $n \in \omega$. 
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4. If \(\alpha \neq \beta\), then the Banach algebra \(\mathcal{B}_n^*(X_\alpha)\) is not linearly isometric (not ring isomorphic etc.) to \(\mathcal{B}_n^*(X_\beta)\) for any \(n \in \omega\).
Decomposition Theorem (K.; Gregoriades and K.; K. and Ng)

If \( f : X \to Y \) is a function from analytic sp. \( X \) into Polish sp. \( Y \) s.t.
\[
A \subseteq \Sigma^0_{m+1}(Y) \Rightarrow f^{-1}[A] \in \Sigma^0_{n+1}(X)
\]
then, there exists a countable partition \( (X_i)_{i \in \omega} \) of \( X \) such that the restriction \( f|_{X_i} \) is \( \Sigma^0_{n-m+1} \)-measurable for every \( i \in \omega \).
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Recursion Theoretic Proof

- By the Louveau separation theorem, we have a Borel measurable transition of a \( \Sigma^0_{m+1} \)-code of \( A \) into a \( \Sigma^0_{n+1} \)-code of \( f^{-1}[A] \).
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- We then have \((f(x) \oplus z)^{(m)} \leq_T (x \oplus (z \oplus p)^{(\xi)})^{(n)}\) for all \( z \in 2^\omega \), where \( \leq_T \) is generalized Turing reducibility on represented spaces.
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- By the **Shore-Slaman join theorem** for any Polish degree structure, we have \( f(x) \leq_T (x \oplus p^{(\xi)})^{(n-m)} \).
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- By the Shore-Slaman join theorem for any Polish degree structure, we have \( f(x) \leq_T (x \oplus p^{(\xi)})^{(n-m)} \).

- Therefore, \( f \) is decomposed into countably many \( \Sigma^0_{n-m+1} \)-measurable functions \( x \mapsto \Phi_e((x \oplus p^{(\xi)})^{(n-m)}), e \in \omega \).
The role of the Decomposition Theorem here is for showing that every \( n \)-th Borel isomorphism is covered by \( \omega \)-many partial homeomorphisms.

\( X \leq_{pw} Y \) means that there is a countable cover \( \{X_i\}_{i \in \omega} \) of \( X \) s.t. \( X_i \) is topologically embedded into \( Y \) for every \( i \in \omega \).

**Main Problem**

Does there exist an uncountable Polish space \( X \) satisfying either of the following equivalent conditions?

1. \( B^*_2(X) \) is linearly isometric neither to \( B^*_2(\mathbb{R}) \) nor to \( B^*_2(\mathbb{R}^N) \).
2. \( B^*_2(X) \) is ring isomorphic neither to \( B^*_2(\mathbb{R}) \) nor to \( B^*_2(\mathbb{R}^N) \).
3. \( X \) is 2\(^{nd} \) level Borel isomorphic neither to \( \mathbb{R} \) nor to \( \mathbb{R}^N \).
4. \( \mathbb{R} \prec_{pw} X \prec_{pw} \mathbb{R}^N \).
Compared to the Borel isomorphism problem in 1970s:

- The *Borel isomorphism problem* on analytic spaces was able to be reduced to the same problem on *zero-dimensional* analytic spaces.
- The *second-level Borel isomorphism problem* is inescapably tied to *infinite dimensional* topology.

Recall: Jayne-Rogers (1979) showed that any two uncountable Polish spaces that are countable unions of sets of finite dimension are 2\textsuperscript{nd}-level Borel isomorphic.

Indeed, Hurewicz-Wallman (1941) showed that

\[ X \cong_{pw} \mathbb{R} \iff \text{trind}(X) < \infty, \]

where \text{trind} is transfinite inductive dimension.
(Alexandrov 1948) $X$ is weakly infinite dimensional (w.i.d.) if for each sequence $(A_i, B_i)$ of pairs of disjoint closed sets in $X$ there are separations $L_i$ in $X$ of $A_i$ and $B_i$ s.t. $\bigcap_i L_i = \emptyset$.

(Haver 1973, Addis-Gresham 1978) $X$ is a $C$-space ($S_c(O, O)$) if for each sequence $(U_i)$ of open covers of $X$ there is a pairwise disjoint open family $(V_i)$ refining $(U_i)$ s.t. $\bigcup_i V_i$ covers $X$.

$$X \leq_{pw} 2^\mathbb{N} \iff \text{trind}(X) < \infty \implies X \text{ is } C \implies X \text{ is w.i.d.}$$

(Alexandrov 1951) $\exists$ a w.i.d. metrizable compactum $X >_{pw} 2^\mathbb{N}$?

(R. Pol 1981) There exists a metrizable $C$-compactum $X >_{pw} 2^\mathbb{N}$.

(E. Pol 1997) There exists an infinite dimensional $C$-Cantor manifold, i.e., a $C$-compactum which cannot be separated by any hereditarily weakly infinite dimensional closed subspaces.

(Chatyrko 1999) There is a collection $\{X_\alpha\}_{\alpha < 2^{\aleph_0}}$ of continuum many infinite dimensional $C$-Cantor manifolds such that $X_\alpha$ cannot be embedded into $X_\beta$ whenever $\alpha \neq \beta$. 

Takayuki Kihara (UC Berkeley)  Second-Level Borel Isomorphism Problem
An infinite dimensional $\mathbf{C}$-Cantor manifold is a $\mathbf{C}$-compactum which cannot be separated by any hereditarily weakly infinite dimensional closed subspace.

**Main Lemma (K. and Pauly)**

Let $\mathcal{M}_\infty$ be the class of all infinite dimensional $\mathbf{C}$-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^{<\omega}, \subseteq)$ into $(\mathcal{M}_\infty, \leq_{pw})$.

- This solves Motto Ros’ problem (and the second level Borel isomorphism problem) in Banach Space Theory.
- This strengthen R. Pol’s theorem and Chatyrko’s theorem in Infinite Dimensional Topology.

To show Main Lemma, we again use Computability Theory!
Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces

(a) Any point in $\mathbb{R}^n$

(b) Some point in $[0, 1]^N$

By approximating each point in a space $X$ by a zero-dim space, we measure "how similar the space $X$ is to a zero-dim space."
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By approximating each point in a space $X$ by a zero-dim space, we measure “how similar the space $X$ is to a zero-dim space”.

(a) Upper and lower approximations by a zero-dim space meet.
(b) There is a gap between upper and lower approximations by a zero-dim space.
Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces

\[ \text{Spec}(x) = \{p \in 2^\mathbb{N} : x \leq_T p\} \]

\[ \text{coSpec}(x) = \{p \in 2^\mathbb{N} : p \leq_T x\} \]

(a) Any point in \( \mathbb{R}^n \)

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- \(\text{Spec}(x) = \{p \in 2^\mathbb{N} : x \leq_T p\}\).
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Key Idea

Classification of topological spaces by degrees of unsolvability:

1. The Turing degrees \( \cong \) the degree structure on Cantor space \( 2^\mathbb{N} \) and Euclidean spaces \( \mathbb{R}^n \).

2. The enumeration degrees \( \cong \) the degree structure on the Scott domain \( \mathcal{P}(\mathbb{N}) \).

3. Hinman (1973): degrees of unsolvability of continuous functionals \( \cong \) the degree structure on the space \( \mathbb{N}^{\mathbb{N}\mathbb{N}} \) of Kleene-Kreisel continuous functionals.

4. J. Miller (2004): continuous degrees \( \cong \) the degree structure on the function space \( C([0, 1]) \) and the Hilbert cube \( [0, 1]^\mathbb{N} \).
Definition

Let $X$ and $Y$ be second-countable $T_0$ spaces with fixed countable open basis $\{B^X_n\}_{n \in \omega}$ and $\{B^Y_n\}_{n \in \omega}$. A point $x \in X$ is "Turing reducible" to a point $y \in Y$ ($x \leq_T y$) if

$$\{n \in \omega : x \in B^X_n\} \leq_e \{n \in \omega : y \in B^Y_n\}.$$ 

In other words, we identify the "Turing degree" of $x \in X$ with the enumeration degree of the (coded) neighborhood filter of $x$.

Example

- The degree structure of **Cantor space** is exactly the same as the **Turing degrees**.
- The degree structure of **Hilbert cube** (a universal Polish space) is exactly the same as the **continuous degrees**.
- The degree structure of **the Scott domain $O(\mathbb{N})$** (a universal quasi-Polish space) is exactly the same as the **enumeration degrees**.
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Spec(\(x\)) = \{p \in 2^\mathbb{N} : x \leq_T p\}; Spec(X) = \{Spec(x) : x \in X\}.
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**Lemma (K. and Pauly)**

$X \simeq_{pw} Y \implies \text{Spec}^r(X) = \text{Spec}^r(Y)$ for some oracle $r \in 2^\omega$.
$\implies \text{coSpec}^r(X) = \text{coSpec}^r(Y)$ for some oracle $r \in 2^\omega$. 

Takayuki Kihara (UC Berkeley)  
Second-Level Borel Isomorphism Problem
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1. A Turing ideal \(\mathcal{I} \subseteq 2^\omega\) is *realized* by \(x\) if \(\mathcal{I} = \text{coSpec}(x)\).
2. A countable set \(\mathcal{I} \subseteq 2^\omega\) is a *Scott ideal* if and only if \((\omega, \mathcal{I}) \models \text{RCA} + \text{WKL}\).

**Realizability of Scott ideals (J. Miller 2004)**

1. \(2^\omega \simeq_{pw} \omega^\omega \simeq_{pw} \mathbb{R}^n \simeq_{pw} \bigoplus_{n \in \omega} \mathbb{R}^n\). (*Turing degrees.*)
   No Scott ideal is realized in these spaces!
2. \([0, 1]^\omega \simeq_{pw} C([0, 1]) \simeq_{pw} \ell^2\). (*full continuous degrees.*)
   Every countable Scott ideal is realized in these spaces!
Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces

![Diagram]

\[ \text{Spec}(x) = \{ p \in 2^\mathbb{N} : x \leq_T p \} \]
\[ \text{coSpec}(x) = \{ p \in 2^\mathbb{N} : p \leq_T x \} \]

(a) Any point in \( \mathbb{R}^n \)
(b) Some point in \( [0, 1]^\mathbb{N} \)

- **Spec** determines the pw-homeomorphism type of a space, and **coSpec** is invariant under pw-homeomorphism.
- The **coSpec** of any point in a space of \( \text{dim} < \infty \) has to be a principal Turing ideal.
- (Miller) Every countable Scott ideal is realized as **coSpec** of a point in Hilbert cube.
Definition

\( \Gamma : 2^\mathbb{N} \to [0, 1]^\mathbb{N} \) is \textit{\( \omega \)-left-CEA operator} if the infinite sequence 
\( \Gamma(y) = (x_0, x_1, x_2, \ldots) \) is generated in a uniformly left-computably enumerable manner by a single Turing machine, that is, there is a left-c.e. operator \( \gamma \) such that for all \( i \),

\[
x_i \coloneqq \Gamma(y)(i) = \gamma(y, i, x_0, x_1, \ldots, x_{i-1}).
\]

An \( \omega \)-left-CEA operator \( \Gamma : \mathbb{N} \times 2^\mathbb{N} \to [0, 1]^\mathbb{N} \) is \textit{universal} if for every \( \omega \)-left-CEA operator \( \Psi \), there is \( e \) such that \( \Psi = \lambda y. \Gamma(e, y) \).
Let $\omega\text{CEA}$ denote the graph of a universal $\omega$-left-CEA operator.

**Theorem (K.-Pauly)**

The space $\omega\text{CEA}$ (as a subspace of Hilbert cube) is an intermediate Polish space:

$$2^\mathbb{N} <_{pw} \omega\text{CEA} <_{pw} [0, 1]^\mathbb{N}$$

**Remark**

Furthermore, $\omega\text{CEA}$ is $pw$-homeomorphic to the following:

- Rubin-Schori-Walsh (1979)’s strongly infinite dimensional totally disconnected Polish space.
- Roman Pol (1981)’s weakly infinite dimensional compactum which is not decomposable into countably many finite-dim subspaces (a solution to Alexandrov’s problem).
(a) $2^\mathbb{N}$  
(b) $\omega$CEA  
(c) $[0, 1]^\mathbb{N}$

- (a) $\text{coSpec}$ is principal, and meets with $\text{Spec}$.
- (b) $\text{coSpec}$ is not always principal, but the “distance” between $\text{Spec}$ and $\text{coSpec}$ has to be at most the $\omega$-th Turing jump.
- (c) $\text{coSpec}$ can realize an arbitrary countable Scott ideal, hence $\text{Spec}$ and $\text{coSpec}$ can be separated by an arbitrary distance.
Proof Sketch of $2^\mathbb{N} <_{pw} \omega\text{CEA} <_{pw} [0, 1]^\mathbb{N}$

\[\omega\text{CEA} = \{(e, p, x_0, x_1, \ldots) \in \omega \times 2^\omega \times [0, 1]^\omega : \]
\[(\forall i) \ x_i \text{ is the } e\text{-th left-c.e. real in } (p, x_0, x_1, \ldots, x_{i-1}).\}\]

**Lemma**

For any $p \in 2^\omega$, the following Scott ideal is not realized in $\omega\text{CEA}$:

\[\mathcal{J}^p = \{z \in 2^\omega : (\exists n) \ z \leq_T p^{(\omega \cdot n)}\}.\]

- Pick $z = (e, p, x_0, x_1, \ldots) \in \omega\text{CEA}$.
- Then, $p \in \text{coSpec}(z)$ and $p^{(\omega)} \in \text{Spec}(z)$.
- Clearly, $p^{(\omega + 1)} \notin \text{coSpec}(z)$.

Since $\text{coSpec}$ (up to an oracle) is invariant under pw-homeomorphism, we have $\omega\text{CEA} <_{pw} [0, 1]^\mathbb{N}$. 
Another separation is based on Kakutani's fixed point theorem.

**Theorem (J. Miller 2004)**

There is a nonempty convex-valued computable function \( \Psi : [0, 1]^\mathbb{N} \rightarrow \mathcal{P}([0, 1]^\mathbb{N}) \) with a closed graph such that for every fixed point \( \langle x_0, x_1, \ldots \rangle \in \text{Fix}(\Psi) \),

\[
\text{coSpec}(\langle x_0, x_1, x_2, \ldots \rangle) = \{x_0, x_1, x_2, \ldots\}.
\]

Moreover, such an \( x \) realizes a Scott ideal.

- \( \text{Fix}(\Psi) \) is a \( \Pi^0_1 \) subset of \([0, 1]^\omega\).
- Inductively find \( (x_0, x_1, \ldots) \in \text{Fix}(\Psi) \), where \( x_{i+1} \) is the "leftmost" value s.t. \( (x_0, x_1, \ldots, x_{i+1}) \) is extendible in \( \text{Fix}(\Psi) \).
- Then, \( x_{i+1} \) is left-c.e. in \( (x_0, x_1, \ldots, x_i) \), uniformly.
- \( x_{i+1} \) does not depend on the choice of a name of \( (x_0, \ldots, x_i) \).
(a) coSpec is principal, and *meets* with Spec.

(b) coSpec is not always principal, but the “distance” between Spec and coSpec has to be at most the \( \omega \)-th Turing jump.

(c) coSpec can realize an arbitrary countable Scott ideal, hence Spec and coSpec can be separated by an arbitrary distance.
**coSpec(2^N) =** all principal Turing ideals.

**coSpec([0, 1]^N) =** all principal Turing ideals and Scott ideals.

3. What do we know about **coSpec(ωCEA)**?
   - It cannot realize an ω-jump ideal.
   - It realizes a non-principal Turing ideal.
   - We know absolutely nothing about what kind of Turing ideals it realizes; even whether it realizes a jump ideal or not.

How can we control **coSpec** of a Polish space?

For instance, given α << β < ω₁, we need a technique for constructing a Polish space such that
   - it cannot realize a β-jump ideal,
   - it realizes an α-jump ideal.
We say that \( g : 2^\mathbb{N} \to 2^\mathbb{N} \) is an oracle \( \Pi^0_2 \) singleton if it has a \( \Pi^0_2 \) graph. For instance, the \( \alpha \)-th Turing jump operator \( T\!J^{\alpha} \) is an oracle \( \Pi^0_2 \) singleton.

**Definition (Modified \( \omega \)CEA Space)**

The space \( \omega \text{CEA}(g) \) consists of \( (d, e, r, x) \in \mathbb{N}^2 \times 2^\mathbb{N} \times [0, 1]^\mathbb{N} \) such that for every \( i \),

1. either \( x_i = g^i(r) \), or

2. there are \( u \leq v \leq i \) such that
   - \( x_i \in [0, 1] \) is the \( e \)-th left-c.e. real in \( \langle r, x_{<i}, x_{l(u)} \rangle \)
   - and \( x_{l(u)} = g^{l(u)}(r) \), where \( l(u) = \Phi_d(u, r, x_{<v}) \).

Here: \( g^0(x) = x \) and \( g^{n+1}(x) = g^n(x) \oplus g(g^n(x)) \).

We define \( \text{Rea}(g) = \omega \text{CEA}(g) \cap (\mathbb{N}^2 \times \text{Fix}(\Psi)) \).

The subspace \( \text{Rea}(g) \) (as a subspace of \( [0, 1]^\mathbb{N} \)) is Polish whenever \( g \) is an oracle \( \Pi^0_2 \) singleton.
Suppose that $g$ is an oracle $\Pi^0_2$-singleton. For every oracle $r \in 2^\mathbb{N}$, consider two Turing ideals defined as

$$J_T(g, r) = \{z \in 2^\mathbb{N} : (\exists n \in \mathbb{N}) \ x \leq_T g^n(r)\},$$
$$J_a(g, r) = \{z \in 2^\mathbb{N} : (\exists n \in \mathbb{N}) \ x \leq_a g^n(r)\}.$$

Here: $\leq_a$ is the arithmetical reducibility.

**Main Lemma (coSpec-Controlling)**

1. For every $x \in \text{Rea}(g)$, there is $r \in 2^\mathbb{N}$ such that
   $$J_T(g, r) \subseteq \text{coSpec}(x) \subseteq J_a(g, r).$$
2. For every $r \in 2^\mathbb{N}$, there is $x \in \text{Rea}(g)$ such that
   $$J_T(g, r) \subseteq \text{coSpec}(x)\)\).$$. 

If $g = TJ^\alpha$ is the $\alpha$-th Turing jump operator for $\alpha \geq \omega$,

1. $\text{coSpec(Rea(TJ^\alpha))}$ realizes no $\beta$-jump ideal for $\beta \geq \alpha \cdot \omega$,
2. $\text{coSpec(Rea(TJ^\alpha))}$ realizes an $\alpha$-jump ideal.
1. By **coSpec**-Controlling Lemma, given an oracle $\Pi^0_2$ singleton $g$, we can construct a Polish space which realizes all Turing ideals closed under $g$.

2. $\text{Rea}(g)$ is strongly infinite dimensional and totally disconnected.

3. Hence, its compactification $\gamma\text{Rea}(g)$ (in the sense of Lelek) is a "Pol-type space", hence, a metrizable $C$-compacta.

4. Note that Lelek's compactification preserves $\text{Spec}$ and $\text{coSpec}$.

5. By combining Elzbieta Pol's construction, our spaces can be assumed to be infinite dimensional $C$-Cantor manifolds.

**Main Lemma (K. and Pauly)**

Let $\mathcal{M}_\infty$ be the class of all infinite dimensional $C$-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^\omega, \subseteq)$ into $(\mathcal{M}_\infty, \leq_{pw})$.
Main Theorem (K. and Pauly)

There exists a $2^\aleph_0$ collection $(X_\alpha)_{\alpha < 2^\aleph_0}$ of topological spaces s.t.

1. $X_\alpha$ is an infinite dimensional Cantor manifold for any $\alpha < 2^\aleph_0$,
2. $X_\alpha$ possesses Haver's property $C$ for any $\alpha < 2^\aleph_0$.
3. If $\alpha \neq \beta$, then $X_\alpha$ is not $n$-th level isomorphic to $X_\beta$ for any $n \in \omega$.
4. If $\alpha \neq \beta$, then the Banach space $B_n(X_\alpha)$ is not linearly isometric to $B_n(X_\beta)$ for any $n \in \omega$.

Summary of This Work

1. Defining the notion of $\text{Spec}$ and $\text{coSpec}$.
2. Using $\text{Spec}$ and $\text{coSpec}$ as "pw-topological" invariant.
3. Proving $\text{coSpec}$-Controlling Lemma.
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