Topological Semantics for Provability Logics

An Overview

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Outline

1. Review of Provability Logic **GL**
2. Polymodal Provability Logic **GLP**
3. The Closed Fragment of **GLP**
4. Ignatiev's Model and the Canonical Model
5. Topological Completeness using ϵ_0
6. Topological Completeness of **GLB**
7. More recent developments

$$\varphi \quad ::= \quad P \;\mid\; \varphi \wedge \varphi \;\mid\; \neg\varphi \;\mid\; \Box\varphi$$

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Let $\mathcal{L}$ be an arithmetical language and T an $\mathcal{L}$-theory with an encoding of a (standard) provability predicate $Prov(\cdot)$, so that

$$\mathcal{N} \models Prov(\ulcorner A \urcorner) \quad \text{iff} \quad T \vdash A\,.$$

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A **realization** is a function from modal to arithmetical formulas:

$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* \quad (\neg\varphi)^* = \neg\varphi^* \quad (\Box\varphi)^* = Prov(\ulcorner \varphi^* \urcorner)\,.$$

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φ is **GL**-valid, if and only if it is PA-provable under all realizations.

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Axioms and rules of **GL**:

1. *Modus ponens* ;
2. Propositional tautologies ;
3. Necessitation: $\frac{\varphi}{\Box\varphi}$;
4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
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Theorem (Segerberg 1971)

GL is sound and complete w.r.t. finite trees.

Consider an increasing sequence of stronger provability predicates:

$$Prov_0 \quad Prov_1 \quad Prov_2 \quad Prov_3 \quad \ldots$$

where $Prov_n(\ulcorner A \urcorner)$ means A is provable from T together with all true Π_n sentences of $\mathcal{L}$ (cf. ω-provable).

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Theorem (Japaridze 1985)

The logic **GLP** is arithmetically complete:

1. **GL** for each modality $[n]$;
2. $[n]\varphi \rightarrow [n+1]\varphi$;
3. $\langle n \rangle\varphi \rightarrow [n+1]\langle n \rangle\varphi$.

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- It turns out $<_0$ has order-type ϵ_0, the 'proof-theoretic ordinal' for PA. (Recall ϵ_0 is the least fixed point of $\omega^\alpha = \beta$.)
- Beklemishev [2004] showed how induction up to ϵ_0, with this algebra providing a **notation system**, can be used to prove consistency of PA.

Frame Incompleteness

Consider a relational frame $\mathbb{F} = (W, \{R_n\}_{n<\omega})$:

(a) $\mathbb{F} \vDash [n]\varphi \to [n+1]\varphi$, iff $R_{n+1} \subseteq R_n$;

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Suppose $\mathbb{F} \vDash$ **GLP** and $R_1 \neq \emptyset$, e.g., xR_1y. Then by (a), xR_0y, and by (b), yR_0y, contradicting converse-well-foundedness of R_0.

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This motivates the search for other classes of models.

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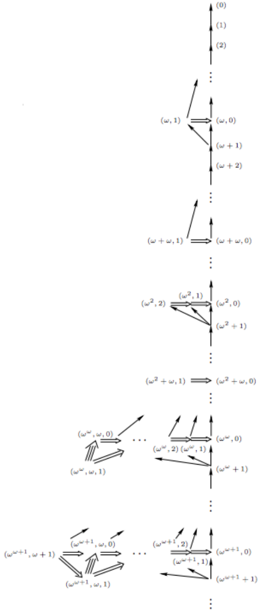
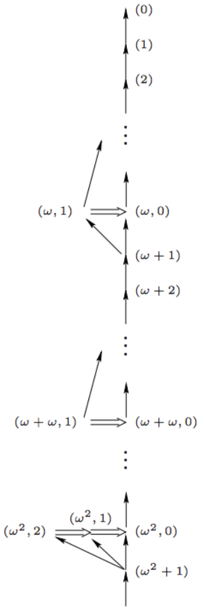
  $$\begin{array}{rcl} U & = & \{\alpha \in \Omega : \alpha_{n+1} \leq l(\alpha_n)\} \\ \alpha R_n \beta & \text{iff} & \alpha_n > \beta_n \text{ and } \alpha_m = \beta_m \text{ for } m < n \ . \end{array}$$



(0)
(1)
(2)
$(\omega, 1)$
$(\omega, 0)$
$(\omega + 1)$
$(\omega + 2)$
$(\omega + \omega, 1)$
$(\omega + \omega, 0)$
$(\omega^2, 1)$
$(\omega^2, 2)$
$(\omega^2, 0)$
$(\omega^2 + 1)$
$(\omega^2 + \omega, 1)$
$(\omega^2 + \omega, 0)$
$(\omega^\omega, \omega, 0)$
$(\omega^\omega, 2)$ $(\omega^\omega, 1)$
$(\omega^\omega, 0)$
$(\omega^\omega, \omega, 1)$
$(\omega^\omega + 1)$
$(\omega^{\omega+1}, \omega, 0)$
$(\omega^{\omega+1}, 2)$
$(\omega^{\omega+1}, \omega + 1)$
$(\omega^{\omega+1}, 0)$
$(\omega^{\omega+1}, 1)$
$(\omega^{\omega+1}, \omega, 1)$
$(\omega^{\omega+1} + 1)$


(0)
(1)
(2)
(ω, 1)
(ω, 0)
(ω + 1)
(ω + 2)
(ω + ω, 1)
(ω + ω, 0)
(ω², 1)
(ω², 2)
(ω², 0)
(ω² + 1)

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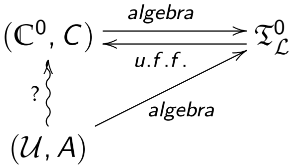

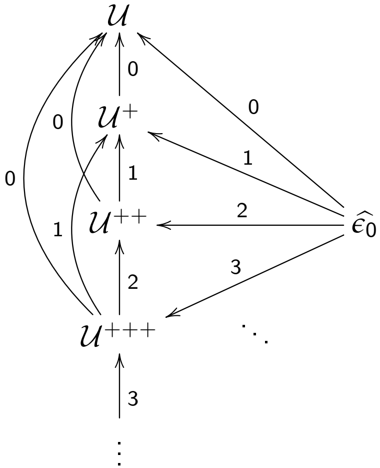
$\mathcal{U}$
$\mathcal{U}^{+}$
$\mathcal{U}^{++}$
$\mathcal{U}^{+++}$
$\widehat{\epsilon_0}$
0
0
0
0
1
1
1
2
2
3
3

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- Esakia also showed **K4** axiomatizes the T_d-spaces.

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Theorem (Abashidze 1985; Blass 1990)

GL is complete with respect to ω^ω with interval topology.

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Theorem (Icard 2011)

GLP^0 is complete with respect to $(\epsilon_0, \{\tau_n\}_{n<\omega})$.

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Theorem (Fernández-Duque and Joosten 2013)

$\mathbf{GLP}^0_\Lambda$ is complete w.r.t. frame and topological semantics.

Definition

A polytopological space $(\mathcal{X}, \{\tau_n\}_{n<\omega})$ is a **GLP**-space if, for all n:

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Given τ_n there is a coarsest τ_{n+1} satisfying the above conditions.

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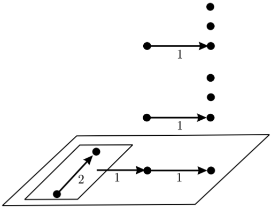

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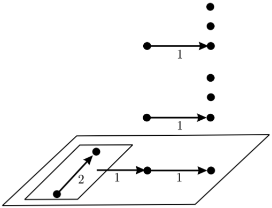

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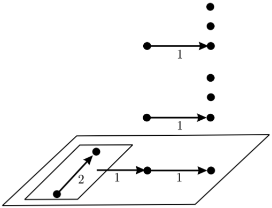

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Ordinal Models

Fact (Beklemishev et al. 2010)

If $(\mathcal{X}, \tau_0, \tau_1)$ is a **GLB**-space and τ_0 is first-countable and Hausdorff, then τ_1 must be discrete.

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- If $cf(\alpha) > \aleph_0$, then any neighborhood of α contains a club.
- Blass [1990] showed incompleteness of **GL** w.r.t. τ_1 is equiconsistent with existence of a weakly Mahlo cardinal.

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- The question of whether there are set-theoretic assumptions that would give ordinal completeness of **GLP** is open.
- Beklemishev and Gabelaia [2013] do show topological completeness of **GLP** w.r.t. general **GLP**-spaces, but (in their own words) "it is not an example of a *natural* **GLP**-space."

Summary

- The study of topological models of provability logic combines work in proof theory, topology, algebra, set theory, and modal logic, exposing some mathematically interesting structures.

Summary

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- It also promises a better grip on the polymodal provability logic **GLP**, providing a simple non-arithmetical interpretation.
- Current work in Moscow and Barcelona is focused on using these models to extend Beklemishev's original ordinal analysis of PA to stronger theories and larger ordinals.

Thanks for your attention!

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