Introduction to Valued Fields

- Consider $\mathbb{R}(t)$, the field of rational functions with real coefficients.
- There is no $\sqrt{-1}$ so this field can be ordered. One way to do this is to say $p(t) < q(t)$ if for all sufficiently large $r \in \mathbb{R}$, $p(r) < q(r)$.
- Say $p(t) \equiv q(t)$ if $p(t) = O(q(t))$ and $q(t) = O(p(t))$. Then $\mathbb{R}(t)^*/\equiv$ is an ordered abelian group, usually called $\Gamma$ and written additively.
- The quotient map $\nu : \mathbb{R}(t)^* \to \Gamma$ is called a valuation.
- The collection of all $p(t)$ in $\mathbb{R}(t)$ such that $p(t) = O(1)$ is a convex ring. This is called the valuation ring, which we will denote $V$.
- The collection of all $m \in V$ such that $1/m \notin V$ forms a maximal ideal, $m$ of $V$. We call these elements infinitessimals.
- The map $\pi : V \to V/m$ is called the standard part map.
Valued Fields in Model Theory

- We will consider fields that are better behaved than $\mathbb{R}(t)$.
- Let $R$ be the real closure of $\mathbb{R}(t)$
  - that is, close $\mathbb{R}(t)$ under square roots of positive elements, and insure that polynomials of odd degree have at least one root.
- We add $\Gamma$ as a sort, as well as $\nu : R^* \rightarrow \Gamma$.
- We add $k = V/m$ as a sort as well as $\pi : V \rightarrow k$.
- This is a real closed valued field and we refer to its theory as RCVF.
- If you form a field extension by adjoining a root of $-1$, you have an algebraically closed valued field and we refer to its theory as ACVF.
Some background

- Haskell, Hrushovski, and Macpherson isolated a phenomena in models of algebraically closed valued fields they called stable domination.
  - A formula, \( \varphi(x, y) \), is stable if it does not have the order property.
    - i.e. there is no \((a_i b_i)_{i < \omega}\) such that \( \varphi(a_i, b_j) \text{ iff } i < j \).
  - Thus valued fields are not stable due to the value group.
  - However, if \( L \) and \( M \) satisfy ACVF and each contain a maximal algebraically closed \( C \) with \( k(L) \) algebraically independent from \( k(M) \) over \( k(C) \) and with \( \Gamma(L) \cap \Gamma(M) = \Gamma(C) \) then \( tp(L/Ck(L)\Gamma(L)) \) implies \( tp(L/M) \).
    - This (roughly) is the property called stable domination.
  - Why “However”? We need a brief detour into stability and independence relations.
What Is An Independence Relation?

• An independence relation, written $A \downarrow^I_C B$, should capture the idea that $B$ and $C$ together contain no additional interesting information about $A$ than $C$ does alone.

• An example: Let $\mathcal{M}$ be a $\mathbb{Q}$-vector space, and let $V$ be a definable subspace of $\mathcal{M}^2$.
  - For instance, let $\mathcal{M} := (\mathbb{R}, +, \{ q \cdot \}_{q \in \mathbb{Q}})$, and let $V$ be the line $q_1 x + q_2 y = 0$.

• Consider two elements of $\mathbb{R}^2$, $a$ and $b$, in the same coset of $V$. Intuitively, $b$ should tell you more about $a$ than you could say without parameters.
  - with the parameter $b$, one can say “$x - b$ is in $V$”. This statement is true of $a$, and if $b_2, b_3, \ldots$ are in different cosets of $V$, then the formulas “$x - b_i$ is in $V$” define pairwise disjoint sets.
  - This is an example of “forking” and one writes $a \nsubseteq b$. 
A Second Example

- Let $\mathcal{M}$ be $(\mathbb{C}, +, \cdot)$
- Consider a tuple, $a$, contained in $\mathbb{C}^3$ not in any algebraic surface defined over $\mathbb{Q}^{alg}$.
  - if there is a surface, defined over $B$, containing $a$ then it seems reasonable to say that $b$ has more information about $a$ than is available over the empty set, and one would write $a \nsubseteq^1 B$
  - Assume there is no curve containing $a$ defined over $B$. If there is a curve containing $a$ defined over $C \supseteq B$, then $a \nsubseteq^1 B$
- i.e. define $A \nsubseteq^1_B C$ to mean there is a tuple of elements of $A$ which is contained over $C$ in a variety of lower dimension than over $B$.
- It turns out that this is not a different independence relation. This is another example of “forking”, and so we write $A \nsubseteq^1_B C$. 
Example

- Let $\mathcal{L} := \{E(x, y)\}$. Let $T$ say that $E$ is an equivalence relation with infinitely many equivalence classes, each of which is infinite.
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• Note that \( xEb_1 \) defines a set which includes \( a \), and the formulas \( xEb_1, xEb_2, xEb_3, \ldots \) define disjoint sets.
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• Note that $xEb_1$ defines a set which includes $a$, and the formulas $xEb_1, xEb_2, xEb_3, \ldots$ define disjoint sets.

$E(x, b_1)$ divides over the empty set, and $\text{tp}(a/b_1)$ forks over the empty set.
Forking

Definition
A formula $\varphi(x, b)$ divides over $C$ if there is $(b_i)_{i \in \mathbb{N}}$ such that
$\{\varphi(x, b_i) | i \in \mathbb{N}\}$ is $k$-inconsistent, and each $b_i \in tp(b/C)$.

Definition
A type forks over $C$ if it implies a disjunction of formulas which divide over $C$.

- Each example of an independence relation so far has been non-forking.
- Non forking is written $a \perp_{b} c$
Unique non-forking extensions

- Non-forking is best behaved in theories that are stable (i.e. no formula has the order property).
- Here one has, among other things, the fact that if $a \downarrow_C B$ and $C$ is a model (or just algebraically closed in $M^{eq}$) then $\text{tp}(a/C)$ implies $\text{tp}(a/BC)$.
- Hence the “however” from many slides ago:
  - However, if $L$ and $M$ satisfy ACVF and each contain a maximal algebraically closed subfield $C$ with $k(L)$ algebraically independent from $k(M)$ over $k(C)$ and with $\Gamma(L) \cap \Gamma(M) = \Gamma(C)$ then $\text{tp}(L/Ck(L)\Gamma(L))$ implies $\text{tp}(L/M)$. 
What goes wrong when there is an order

- Let $\mathcal{L} := \{<\}$. Let $T$ be the theory of dense linear orders.
- $(\mathbb{Q}, <)$
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\[ a_0 < x < b_0 \quad (a_1 < x < b_1) \quad (a_2 < x < b_2) \]
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(c_0 < x < d_0) & \\
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So $a_0 < x < b_0$ divides over $\{a_0, b_0\}$. 
What goes wrong when there is an order

- Let $\mathcal{L} := \{<\}$. Let $T$ be the theory of dense linear orders.
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So $c_0 < x < d_0$ divides over $\{a_0, b_0\}$. There is no end to the “information” that you can have about an element. So forking is not an independence relation.
\[ \mathfrak{b}\text{-Forking} \]

- There is a generalization of stability (and of simplicity), called *rosiness* that is not ruined by the existence of an order.
- We want a definition similar to forking but that is well-behaved in a larger variety of settings.

**Definition**

A formula \( \varphi(x, b) \) \( \mathfrak{b}\)-divides over \( C \) if there is some \( \theta(y, d) \) such that \( \{ \varphi(x, b) | b \models \theta(y, d) \} \) is \( k \)-inconsistent, and \( \text{tp}(b/Cd) \) is infinite and contains \( \theta(y, d) \).

**Definition**

A type \( \mathfrak{b}\text{-forks} \) over \( C \) if it implies a disjunction of formulas which \( \mathfrak{b}\)-divide over \( C \).

- \( \mathfrak{b}\)-forking is a more uniform version of forking.
- One writes \( a \downarrow^\mathfrak{b}_B C \) to indicate non-\( \mathfrak{b}\)-forking.
Back to $(\mathbb{Q}, <)$
Back to $(\mathbb{Q}, <)$
Back to \((\mathbb{Q}, <)\)
Back to \((\mathbb{Q}, <)\)
So $a_0 < x < b_0$ does not $\beta$-divide. (Only things of the form $x = b$ $\beta$-divide.)
So $a_0 < x < b_0$ does not $\exists$-divide. (Only things of the form $x = b$ $\exists$-divide.)
Back to \((\mathbb{Q}, <)\)

- So \(a_0 < x < b_0\) does not \(\mathfrak{b}\)-divide. (Only things of the form \(x = b\) \(\mathfrak{b}\)-divide.)

**Definition**

When \(\independ^b\) is an independence relation on \(\mathcal{M}^{eq}\), we call the theory \emph{rosy}.

- Note in this example if \(a \independ_C b\), \(tp(a/C)\) does not imply \(tp(a/Cb)\)
There is no \( b \)-dividing! \( xEb_1 \) and \( xEb_2 \) define disjoint sets, but \( xEb_1 \) and \( xEb \) define identical sets.
• There is no $\mathfrak{b}$-dividing!
• But the problem would be solved if we could treated $a/E$ as an element. Then the formula “$x$ is in the equivalence class $a/E$” would $\mathfrak{b}$-divide.
• When one adds to $\mathcal{M}$ sorts for quotients of definable equivalence relations, one forms $\mathcal{M}^{eq}$.
• Working in $\mathcal{M}^{eq}$ in a stable theory, forking and $\mathfrak{b}$-forking coincide.
• In any theory, non-forking is the strongest independence relation and non-$\mathfrak{b}$-forking is the weakest.
Back to ACVF

- Just as forking “over reacts” to the presence of an order, \( \pi \)-forking “over reacts” to the presence of an ultrametric.
- And \( \pi \)-forking independence is the weakest possible independence relation, so ACVF does not admit any independence relation.
- However, when one has \( C, L, M \) with \( k(L) \downarrow_C k(M) \) and \( \Gamma(L) \downarrow_C \Gamma(M) \) then when \( C \) is maximal and algebraically closed, \( tp(L/Ck(L)\Gamma(L)) \) implies \( tp(L/M) \).
- Philosophy: Once one controls for the value group, ACVF is one stable structure sitting on top of another one.
Residue Field Domination

• Idea: After accounting for the value group, a real closed valued field is an o-minimal structure sitting on top of another o-minimal structure.

• Guess: If $C \models RCVF$ be a maximal field which is a submodel of both $L$ and $M$, and suppose that $k(L)\Gamma(L) \downarrow^b_C k(M)\Gamma(M)$ then $tp(L/Ck(L)\Gamma(L))$ together with $tp < (L/M)$ implies $tp(L/M)$.

Theorem (E., Haskell, Maříková)

In fact, $tp(L/Ck(L)\Gamma(L))$ implies $tp(L/M)$.

Theorem (E., Haskell, Maříková)

In either $RCVF$ or $ACVF$. Suppose $C$ is maximal and a model. Then

i) $a \downarrow^b_C b$ if and only if $k(Ca)\Gamma(Ca) \downarrow^b_C k(Cb)\Gamma(Cb),$

ii) $a \downarrow_C b$ if and only if $k(Ca)\Gamma(Ca) \downarrow_C k(Cb)\Gamma(Cb).$