The algebra of topology: Tarski's program 70 years later

Guram Bezhanishvili
New Mexico State University

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- As a consequence of the two representation theorems, they proved that Gödel's translation is full and faithful.

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To see this, it is convenient to first discuss representation of closure algebras and Heyting algebras. These representations generalize the celebrated Stone representation of Boolean algebras.

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Furthermore, B is isomorphic to the Boolean algebra of clopens (= closed and open sets) of this topology.

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McKinsey-Tarski representation: Every closure algebra can be represented as a subalgebra of the closure algebra $(\wp(X), \mathbf{cl})$ for some topological space X.

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Consequently, every Heyting algebra can be represented as a subalgebra of the Heyting algebra of opens of some topological space.

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Gödel-McKinsey-Tarski Theorem: $\mathbf{IPC} \vdash \varphi$ iff $\mathbf{S4} \vdash \varphi^t$.

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Since R is a preorder, it gives rise to the Alexandroff topology τ_R on X, where the closure of $U \subseteq X$ is given by

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Jónsson-Tarski (1951), Kripke (1963): Every closure algebra can be represented as a subalgebra of the closure algebra $(\wp(X), R^{-1})$ for some preordered set (X, R).

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Extensions of **S4** have unique intuitionistic fragments, but extensions of **IPC** have many modal companions.

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The Blok-Esakia theorem (1976): The lattice of extensions of **IPC** is isomorphic to the lattice of extensions of **Grz**.

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The expressive power can be further extended by introducing nominals. But this may lead to undecidability of our system. One direction of current research is to seek a good balance between expressive power and decidability of a modal system.

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Surprisingly, this question has a positive solution. For example, we can pick up every extension of **S4** from subalgebras of the closure algebra of Cantor's discontinuum! This is no longer so if we work with the real line (connectedness gets in the way). Nevertheless, it is possible to describe the logics that arise as logics of subalgebras of the closure algebra of the real line.

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New results in measure-theoretic interpretation of modal logic are being proved as we speak!

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We do have an adequate semantics for one-variable fragments of these systems by means of monadic Heyting algebras and monadic modal algebras. But in its general form, the Blok-Esakia theorem remains unsolved even for these weaker systems

First-order logics

The Gödel translation of **IPC** into **S4** extends to the predicate case. However, it remains an open problem whether there is a predicate analogue of the Blok-Esakia theorem.

One of the difficulties is the lack of adequate semantics in the predicate case. While both first-order intuitionistic logic and first-order **S4** are complete (algebraically, topologically, or relationally), this is no longer true for many extensions of these logics. Thus, it is desirable to obtain a workable adequate semantics of these systems. Some attempts in this direction include sheaf semantics, and more generally, bundle semantics.

We do have an adequate semantics for one-variable fragments of these systems by means of monadic Heyting algebras and monadic modal algebras. But in its general form, the Blok-Esakia theorem remains unsolved even for these weaker systems (some partial results in this direction are available).

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Thank you!