Contextuality, Cohomology and Paradox

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- Cohomology. Sheaf theory provides the natural mathematical setting for our analysis, since it is directly concerned with the passage from local to global. In this setting, it is furthermore natural to use **sheaf cohomology** to characterise contextuality. Cohomology is one of the major tools of modern mathematics, which has until now largely been conspicuous by its **absence**, in logic, theoretical computer science, and quantum information.

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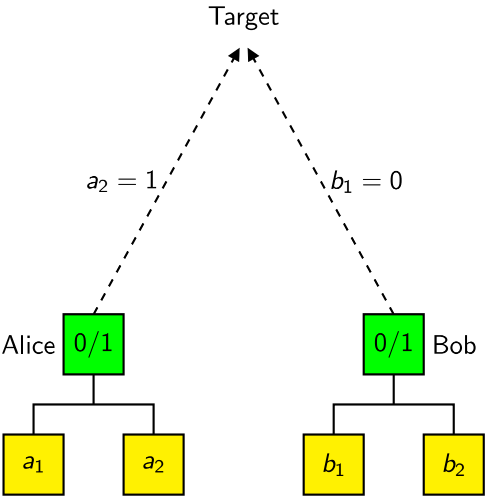
- Our results show that **cohomological obstructions to the extension of local sections to global ones** witness a large class of contextuality arguments.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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a_2	b_1	3/8	1/8	1/8	3/8
a_2	b_2	1/8	3/8	3/8	1/8

The entry in row 2 column 3 says:

If Alice looks at a_1 and Bob looks at b_2, then 1/8th of the time, Alice sees a 0 and Bob sees a 1.

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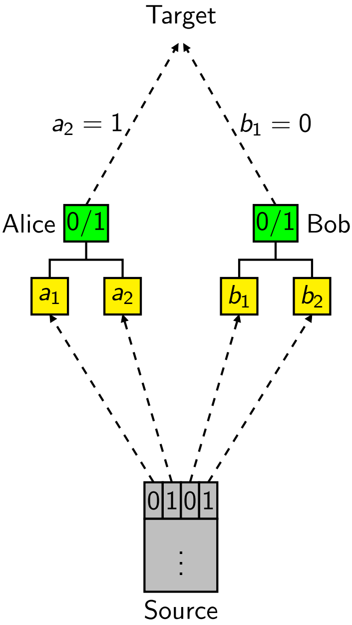
How can we explain this behaviour?

Classical Correlations: The Classical Source



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Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathrm{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathrm{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1}(1-p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \leq N - 1.$$

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(a_1, b_1)	$1/2$	0	0	$1/2$
(a_1, b_2)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2, b_1)	$3/8$	$1/8$	$1/8$	$3/8$
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If we read 0 as true and 1 as false, the highlighted entries in each row of the table are represented by the following propositions:

$$
\begin{aligned}
\varphi_1 &= (a_1 \wedge b_1) \vee (\neg a_1 \wedge \neg b_1) = a_1 \leftrightarrow b_1 \\
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The violation of the logical Bell inequality is $1/4$.

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The support of the Hardy model:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
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Hence the Hardy model achieves a violation of $p_1 = \text{Prob}(a \wedge b)$ for the logical Bell inequality.

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Can we explain this behaviour using a classical source?

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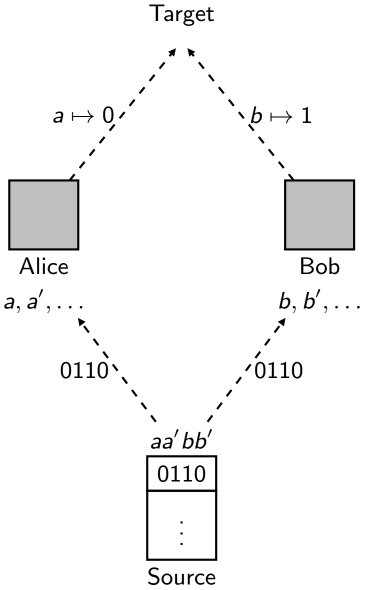
This point of view is called **non-contextuality**. It is equivalent to the assumption of a classical source.

However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



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Hardy models: those whose support satisfies

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Thus Hardy models are **contextual**. They cannot be explained by a classical source.

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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than "which register to load".

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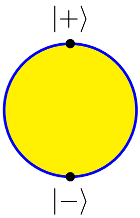
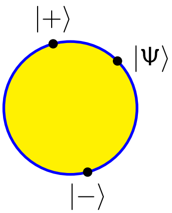
States of the system can be described by complex unit vectors in $\mathbb{C}^2$. These can be visualized as points on the unit 2-sphere:





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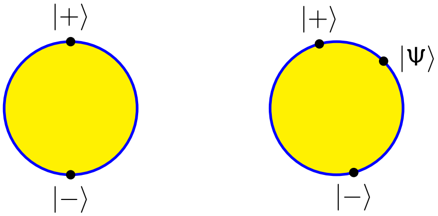

Spin can be measured in any direction; so there are a continuum of possible measurements. There are **two possible outcomes** for each such measurement; spin in the specified direction, or in the opposite direction. These two directions are represented by a pair of orthogonal vectors. They are represented on the sphere as a pair of **antipodal points**.

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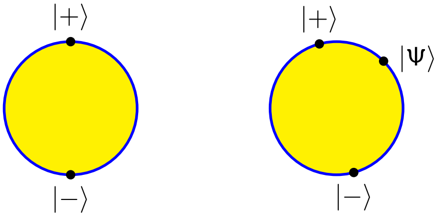

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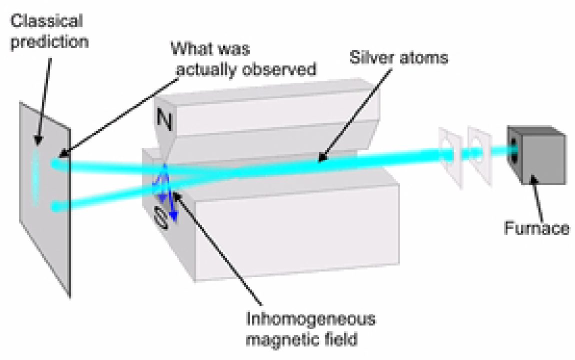
Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



Quantum Entanglement

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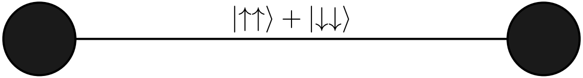

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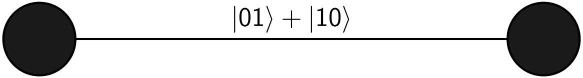

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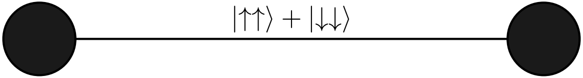

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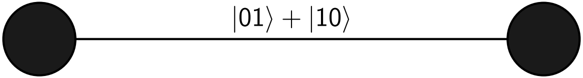

Compound systems are represented by **tensor product**: $\mathcal{H}_1 \otimes \mathcal{H}_2$. Typical element:

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Superposition encodes **correlation**.

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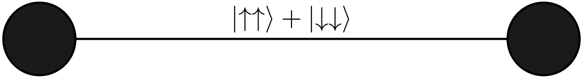

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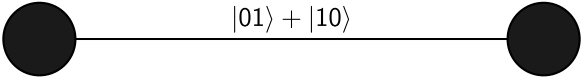

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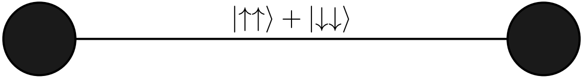

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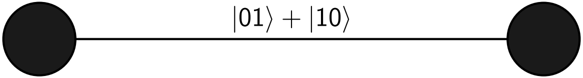

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Bell's theorem: QM is **essentially non-local**.

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	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
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This proves a **strong version of Bell's theorem**.

Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
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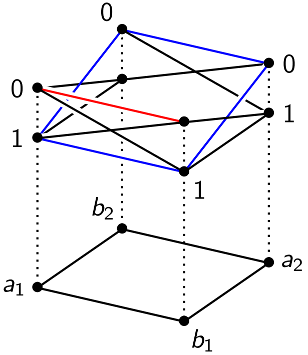
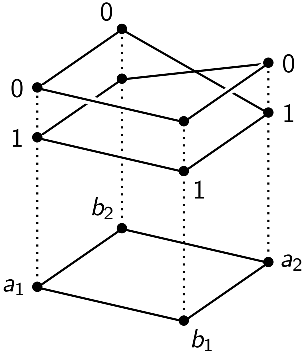
The PR Box

Visualizing Contextuality





The Hardy table and the PR box as bundles

Contextuality, Logic and Paradoxes

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Liar cycles. A Liar cycle of length N is a sequence of statements

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The “paradoxical” nature of the original statements is now captured by the inconsistency of these equations.

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We can regard each of these equations as fibered over the set of variables which occur in it:

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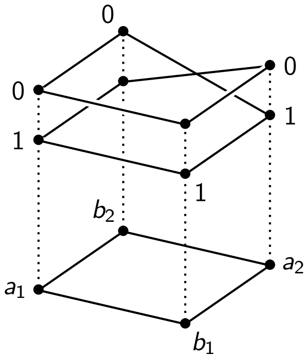
The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



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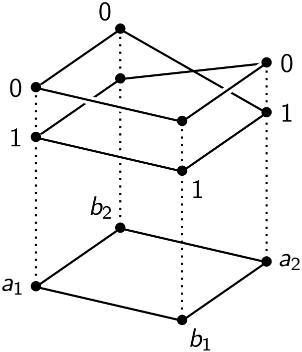

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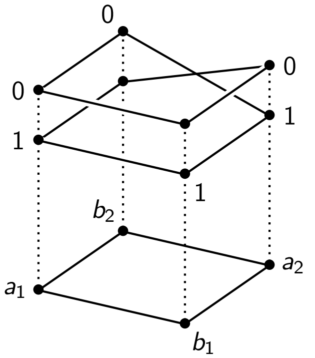

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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

The Robinson Consistency Theorem

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A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

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This example is well-known in the quantum contextuality literature as the **Specker triangle**.

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- O is set of outcomes or values for the variables, which we take to be the same in each fibre.

We have a sheaf of sets over $\mathcal{P}(X)$, namely $\mathcal{E} :: U \longmapsto O^U$ with restriction

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A probability table can be represented by a family $\{p_C\}_{C \in \mathcal{M}}$ with p_C a probability distribution on $\mathcal{E}(C) = O^C$, where contexts C corresponds to the rows of the table.

Empirical Models

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The logical and strong forms of contextuality are concerned with **possibilities**, which can be represented by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each context $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible outcomes.

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Explicitly, $\mathcal{S}$ is defined as follows, where $\mathsf{supp}\,(p_C|U \cap C)$ is the support of the marginal of p_C at $U \cap C$.

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We can use this formalisation to characterize contextuality as follows.

Definition

For any empirical model $\mathcal{S}$:

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\mathcal{S}$ is logically contextual at s, written $\mathsf{LC}(\mathcal{S}, s)$, if s is not a member of any compatible family.
- $\mathcal{S}$ is **strongly contextual**, written $\mathsf{SC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for all s. Equivalently, if it has no global section, *i.e.* if $\mathcal{S}(X) = \varnothing$.

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We define

$$\gamma(s) = [z] \in \check{H}^1(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$$

where $\mathcal{F}$ is the **AbGrp**-valued presheaf $\mathbb{Z}[S_e]$.

Summary of Cohomological Characterization

We have a cover

$$\mathcal{U} = \{C_1, \ldots, C_n\}$$

of measurement contexts.

Given $s = s_1 \in S_e(C_1)$, we define

$$z = \delta^0(s_1, \ldots, s_n),$$

where $s_1|_{C_1 \cap C_i} = s_i|_{C_1 \cap C_i}$, $i = 1, \ldots, n$.

This is a cocycle in the relative Čech cohomology with respect to C_1.

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Here γ is in fact the **connecting homomorphism** of the long exact sequence.

Basic Results

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Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a family* $\{r_i \in \mathcal{F}(C_i)\}$ *with* $s_1 = r_1$, *and for all* i, j:

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Proposition

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Thus non-vanishing of the obstruction provides a cohomological witness for contextuality.

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- In recent work, we obtain very general results in cases where the outcomes themselves have a module structure (over the same ring as the cohomology coefficients).
- This yields cohomological characterisations of **All-vs.-Nothing** proofs (Mermin). These account for most of the contextuality arguments in the quantum literature. In particular, we can find large classes of concrete examples in **stabiliser QM**.

Theorem

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R \rangle$. Then:

$$\mathsf{AvN}_R(\mathcal{S}) \Rightarrow \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}) \Rightarrow \mathsf{CSC}_R(\mathcal{S}) \Rightarrow \mathsf{CSC}_{\mathbb{Z}}(\mathcal{S}) \Rightarrow \mathsf{SC}(\mathcal{S}) .$$

Relational databases

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Hanover	10992-35671	Leibniz	€11,245.75
...	...	...	...

From possibility models to databases

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Consider again the Hardy model:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
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Change of perspective:

$a_1,\ a_2,\ b_1,\ b_2$	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
1	1

a_1	b_2
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There is no **universal relation**: no table

a_1	a_2	b_1	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

A dictionary

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Relational databases	measurement scenarios
attribute	measurement
set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
relation/set of tuples	boolean distribution on joint outcomes
universal relation instance	global section/hidden variable model
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We can also consider probabilistic databases and other generalisations; cf. provenance semirings.

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For an accessible overview of Contextual Semantics, see the article in the *Logic in Computer Science* Column, Bulletin of EATCS No. 113, June 2014 (and `arXiv`).

People

Comrades in Arms in Contextual Semantics:

People

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People

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Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida

Some Recent Developments

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- **Hardy is almost everywhere**: with bipartite exceptions, an algorithm which given an n-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.
- Characterization of the **face lattice** of the No-Signalling polytope as isomorphic to the support lattice.
- General characterisation of **All-versus-Nothing** arguments. The cohomology invariant captures contextuality for all such models. Large classes of quantum examples using stabiliser groups.

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