Degrees that are not Degrees of Categoricity

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A structure (coded as a subset of $\omega$) is a **computable structure** if its domain and atomic diagram are computable.

Without loss of generality, we assume all computable structures have domain $\omega$.

We denote the $n$-th computable structure under some effective listing by $A_n$. 
**Definition**

Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is **computably categorical** if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$.

**Example**

Given two computable copies of the dense linear orders without endpoints (DLO) we can find a computable isomorphism between them. Therefore they are computably categorical structures.
Computably categorical structures

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**Example**

Given two computable copies of the dense linear orders without endpoints (DLO) we can find a computable isomorphism between them.

Therefore they are computably categorical structures.
Definition

Let $\mathcal{A}$ be a computable structure and $x$ a Turing degree. We say that $\mathcal{A}$ is $x$-computably categorical if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T x$. 

Example

The standard ordering on $\mathbb{N}$ is $0'$-computably categorical. To build an isomorphism to a computable copy, we use $0'$ to determine how many predecessors each element has.
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Example
The standard ordering on $\mathbb{N}$ is $\mathbf{0}'$-computably categorical.

To build an isomorphism to a computable copy, we use $\mathbf{0}'$ to determine how many predecessors each element has.
Degrees of categoricity

**Definition**

CatSpec(\(\mathcal{A}\)) = \{x \mid \mathcal{A} \text{ is } x\text{-computably categorical}\}

**Definition (Fokina, Kalimullin, and Miller)**

A Turing degree \(x\) is a degree of categoricity if there is a computable structure \(\mathcal{A}\) such that \(x \in \text{CatSpec}(\mathcal{A})\) and for all \(y \in \text{CatSpec}(\mathcal{A})\) we have \(x \leq_T y\).

Degrees of categoricity are sometimes called categorically definable degrees.
Degrees of categoricity

Definition

\[ \text{CatSpec}(\mathcal{A}) = \{ x \mid \mathcal{A} \text{ is } x\text{-computably categorical} \} \]

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Summary

A witnesses \( x \) is a degree of categoricity if \( x \) is the least degree that can compute isomorphisms between \( A \) and any computable structure isomorphic to it.

Example

For example, computable copies of the DLO witness that \( 0 \) is a degree of categoricity.
Strong degrees of categoricity

**Definition**

A Turing degree $x$ is a **strong degree of categoricity** if there is a computable structure $A$ with computable copies $B$ and $M$ such that $A$ is $x$-computably categorical, and for every isomorphism $f : B \to M$ we have $x \leq_T f$.

**Remark**

*Strong degrees of categoricity are degrees of categoricity.*
Known results (positive)

Fokina, Kalimullin, and Miller developed the basic method for showing degrees are degrees of categoricity.

**Theorem (Fokina, Kalimullin, and Miller)**

Let $x$ be a d.c.e. degree. Then $x$ is a [strong] degree of categoricity.

This result can be relativized to finite and transfinite jumps.

**Theorem (Fokina, Kalimullin, and Miller)**

Let $n \in \omega$ and let $x$ be d.c.e.($\emptyset$(n)) with $x \geq T_\emptyset(n)$. Then $x$ is a [strong] degree of categoricity.

**Theorem (Csima, Franklin, and Shore)**

Let $\alpha < \omega_{CK}^1$ and let $x$ be d.c.e.($\emptyset$(\alpha)) with $x \geq T_\emptyset(\alpha)$. Then $x$ is a [strong] degree of categoricity.
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**Theorem (Csima, Franklin, and Shore)**

Let $\alpha < \omega_1^{CK}$ and let $x$ be d.c.e.($\emptyset^{(\alpha)}$) with $x \geq_T \emptyset^{(\alpha)}$. Then $x$ is a [strong] degree of categoricity.
It is easy to see that there are at most countably many degrees of categoricity.

It has been shown that all degrees of categoricity are hyperarithmetic.

**Theorem (Fokina, Kalimullin, and Miller)**

If $x \notin \text{HYP}$, then $x$ is not a strong degree of categoricity.

**Theorem (Csima, Franklin, and Shore)**

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In this talk we will show several more negative results. We start by considering a straight-forward example.

**Proposition (Anderson and Csima)**

There is a degree $x \leq_T 0''$ that is not a degree of categoricity.
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**Proposition (Anderson and Csima)**

There is a degree $x \leq_T 0''$ that is not a degree of categoricity.

**Ideas for proof**

- We construct a noncomputable $X$ by finite extensions using a $\emptyset''$ oracle.

- We build $X$ so that for any computable structure $A_m$ we have $\text{Deg}(X) \in \text{CatSpec}(A_m) \Rightarrow 0 \in \text{CatSpec}(A_m)$. 
For every \((l, m, k)\) we want to satisfy:

Either \(\Phi^X_l\) is not an isomorphism from \(A_m\) to \(A_k\), or there is a computable isomorphism.
Warm up proposition (continued)

Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy: Either $\Phi_l^X$ is not an isomorphism from $A_m$ to $A_k$, or there is a computable isomorphism.

- Given a string $\sigma$ we wish to determine if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ cannot be extended to an isomorphism.
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- Given a string \(\sigma\) we wish to determine if there is a \(\tau \supseteq \sigma\) such that \(\Phi^\tau_l\) cannot be extended to an isomorphism.

- We ask \(\emptyset'\): Is there a \(\tau \supseteq \sigma\) such that \(\Phi^\tau_l\) is seen not to be an injective homomorphism?
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- We ask \(\emptyset'\): Is there a \(\tau \supseteq \sigma\) such that \(\Phi_l^\tau\) is seen not to be an injective homomorphism?

- We ask \(\emptyset''\): Is there a \(\tau \supseteq \sigma\) and a \(d \in \omega\) such that for every \(\gamma \supseteq \tau\) we have \(d\) is not in the domain or range of \(\Phi_l^\gamma\)?
For every \((l, m, k)\) we want to satisfy:
Either \(\Phi^X_l\) is not an isomorphism from \(A_m\) to \(A_k\), or there is a computable isomorphism.

Given a string \(\sigma\) we wish to determine if there is a \(\tau \supseteq \sigma\) such that \(\Phi^\tau_l\) cannot be extended to an isomorphism.

We ask \(\emptyset'\): Is there a \(\tau \supseteq \sigma\) such that \(\Phi^\tau_l\) is seen not to be an injective homomorphism?

We ask \(\emptyset''\): Is there a \(\tau \supseteq \sigma\) and a \(d \in \omega\) such that for every \(\gamma \supseteq \tau\) we have \(d\) is not in the domain or range of \(\Phi^\gamma_l\)?

Yes: extend to \(\tau\). No: there is a computable isomorphism.
We wish to generalize this proof to come up with a negative result on a broad class of sets.

**Definition**

A set $G$ is **$n$-generic** if for every $\Sigma_n$ subset $S$ of $2^{<\omega}$ there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ we have $\tau \notin S$. 

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2-generic relative to some perfect tree

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**Definition**

A set $G$ is \(n\)-generic relative to the perfect tree $T$ if $G$ is a path through $T$ and for every $\Sigma_n(T)$ subset $S$ of $T$, there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ with $\tau \in T$ we have $\tau \notin S$. 

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**Definition**

A set $G$ is **$n$-generic relative to some perfect tree** if there exists a perfect tree $T$ such that $G$ is $n$-generic relative to $T$. 
We can now use this to limit degrees of categoricity to a small, easily defined class.

**Theorem (Anderson)**

For every $n$, there are only countably many sets that are not $n$-generic relative to any perfect tree.

**Theorem (Anderson and Csima)**

Let $G$ be 2-generic relative to some perfect tree and $g = \text{Deg}(G)$. Then $g$ is not a degree of categoricity.
We can now use this to limit degrees of categoricity to a small, easily defined class.

**Theorem (Anderson)**

For every $n$, there are only countably many sets that are not $n$-generic relative to any perfect tree.

Generalizing the methods used to construct a degree below $0''$ we can prove:

**Theorem (Anderson and Csima)**

Let $G$ be 2-generic relative to some perfect tree and $g = \text{Deg}(G)$. Then $g$ is not a degree of categoricity.
The theorem allows us to find a degree that is not a degree of categoricity between any set and its double jump.

**Corollary**

Let $X$ and $A$ be sets such that $X$ is 2-generic (A). Then $x \oplus a$ is not a degree of categoricity.

**Corollary**

For every $x$ there is a $y$ such that $x \leq_T y \leq_T x''$ and $y$ is not a degree of categoricity.
We can also exclude degrees of categoricity from another class.

**Definition**

A degree $x$ is hyperimmune-free if for every function $f \leq_T x$ there is a computable function $g$ which dominates $f$.

We notice that all known degrees of categoricity are between jumps and hence hyperimmune.
We can also exclude degrees of categoricity from another class.

**Definition**

A degree $x$ is **hyperimmune-free** if for every function $f \leq_T x$ there is a computable function $g$ which dominates $f$.

We notice that all known degrees of categoricity are between jumps and hence hyperimmune.

**Theorem (Anderson and Csima)**

Let $x$ be a noncomputable hyperimmune-free degree. Then $x$ is not a degree of categoricity.
There are no hyperimmune-free degrees or degrees of sets 2-generic relative to some perfect tree that are $\Sigma_2$.

However, we can construct a $\Sigma_2$ set whose degree is not a degree of categoricity directly.

**Theorem (Anderson and Csima)**

*There is a $\Sigma_2$ degree that is not a degree of categoricity.*
Ideas for proof

- We construct $X$ to be c.e. in a $\emptyset'$ oracle.
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- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.

- We weaken the requirement that $x \in \text{CatSpec}(\mathcal{A}_m) \Rightarrow 0 \in \text{CatSpec}(\mathcal{A}_m)$.

- Instead, for each $m \in \omega$ we construct a $Y_m \not\leq_T X$ such that for all $k$, if $X$ computes an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ then so does $Y_m$. 

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- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.

- We weaken the requirement that $x \in \text{CatSpec}(A_m) \Rightarrow 0 \in \text{CatSpec}(A_m)$.

- Instead, for each $m \in \omega$ we construct a $Y_m \not\geq^T X$ such that for all $k$, if $X$ computes an isomorphism from $A_m$ to $A_k$ then so does $Y_m$.

- Each $Y_m$ witnesses $x$ is not the least degree in $\text{CatSpec}(A_m)$. 
Ideas for proof (continued)

- We split each $Y_m$ into columns, $Y_m^{[l,k]}$.

- We maintain $Y_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all $t$. 
We split each $\gamma_m$ into columns, $\gamma_m^{[l,k]}$.

We maintain $\gamma_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all $t$.

If we appear unable to block $\Phi_l^X$ from becoming an isomorphism from $A_m$ to $A_k$, we will try to make $f = \Phi_l \gamma_m^{[l,k]}$ an isomorphism.
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If we appear unable to block $\Phi_l^X$ from becoming an isomorphism from $A_m$ to $A_k$, we will try to make $f = \Phi_l^{Y_m^{[l,k]}}$ an isomorphism.

We build $X$ by finite extensions except at special stages called slides.
### Ideas for proof (continued)

- Given $\sigma$ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that $\Phi_{l}^{\tau}$ is not a partial injective homomorphism from $A_m$ to $A_k$.

- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_{m}^{[l,k]} \upharpoonright \sigma$. 
Given $\sigma$ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that $\Phi^\tau_l$ is not a partial injective homomorphism from $A_m$ to $A_k$.

At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_m^{[l,k]} \upharpoonright \sigma$.

If yes, we extend to $\tau$ and are done for $(l,m,k)$.

If no, then for all $\gamma \supseteq \sigma$ we have $\Phi^\gamma_l$ is a partial injective homomorphism.
Ideas for proof (continued)

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- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_{m}^{[l,k]} \upharpoonright \sigma$.

- If yes, we extend to $\tau$ and are done for $(l, m, k)$.

- If no, then for all $\gamma \supseteq \sigma$ we have $\Phi^\gamma_l$ is a partial injective homomorphism.

- We attempt to build $Y_{m}^{[l,k]} \supseteq \sigma$ by finite extensions to ensure every $d \in \omega$ is in the domain and range of $f = \Phi^Y_{m}^{[l,k]}$. 

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Problem: What if no extension for $Y_{m}^{[l,k]}$ puts $d$ into the domain and range of $f$?

In this case we perform a slide. We change $X(t)$ from 0 to 1 for all $t$ where $X$ differs from $Y_{m}^{[l,k]}$. We now have $X = Y_{m}^{[l,k]}$ and since $\Phi_X$ cannot be made into an isomorphism, we are done for $(l, m, k)$. Many weaker priorities are injured, but a finite injury construction is possible.
### Ideas for proof (conclusion)

- **Problem:** What if no extension for $\gamma_m^{[l,k]}$ puts $d$ into the domain and range of $f$?

- In this case we perform a slide. We change $X(t)$ from 0 to 1 for all $t$ where $X$ differs from $\gamma_m^{[l,k]}$.

- We now have $X = \gamma_m^{[l,k]}$ and since $\Phi_l^X$ cannot be made into an isomorphism, we are done for $(l, m, k)$. 

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Ideas for proof (conclusion)

- Problem: What if no extension for $\gamma_{l,k}^m$ puts $d$ into the domain and range of $f$?

- In this case we perform a slide. We change $X(t)$ from 0 to 1 for all $t$ where $X$ differs from $\gamma_{l,k}^m$.

- We now have $X = \gamma_{l,k}^m$ and since $\Phi_i^X$ cannot be made into an isomorphism, we are done for $(l, m, k)$.

- Many weaker priorities are injured, but a finite injury construction is possible.
Conclusion

There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

Open questions

1. Is every 3-c.e. degree a degree of categoricity?
2. Is there a $\Delta^2_2$ degree which is not a degree of categoricity?
3. Is there a degree of categoricity which is not a strong degree of categoricity?

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